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An useful analytical formula to avoid thermal damage in the adaptive control of dry surface grinding

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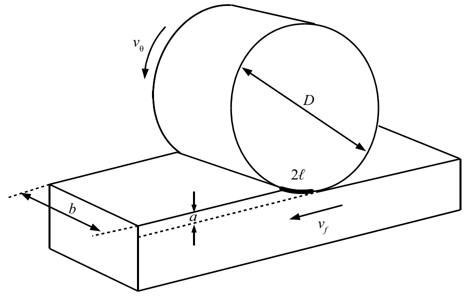
Abstract

An adaptive control is proposed for dry surface grinding to extend the use of the wheel without needing to be dressed, preserving at the same time the surface integrity of the workpiece. The implementation of this adaptive control needs to use predictive models of thermal damage, as in the case of Malkin's model, which calculates the allowable grinding power before the workpiece gets burnt for any working condition. In this latter case, the adaptive control of the cutting depth condition requires solving a quartic equation. Since the analytical procedures for solving quartics given in the literature are quite cumbersome to implement in the numeric control of the grinding machine, we propose a closed analytic formula in order to compute directly the unique positive solution. Moreover, we can enhance Malkin's model in order to consider an arbitrary heat flux profile entering into the workpiece and the kinematical correction to the geometrical contact length, in such a way that we can still using the latter solution to the quartic equation.

1 Introduction

Surface grinding is a industrial machining process used to polish the surface of a workpiece by means of a high speed rotating wheel which removes the



workpiece material by abrasion. In Fig. 1, kinematical and geometrical parameters of this machining process are shown. As kinematical parameters, we have the peripheral velocity of the wheel v_θ ($\mathrm{m\,s^{-1}}$ in SI units) and its feedrate v_f. As geometrical parameters, we have the contact length between wheel and workpiece 2ℓ (m), the diameter of the wheel D, the workpiece width that is being ground b and the cutting depth a. Since most of the energy consumed in the grinding process is converted into heat by friction, the principal problem of this process is the thermal damage of the workpiece. For instance, burning occurs when the maximum temperature reached in the workpiece $T_{\max}$ (K) exceeds the change of phase temperature of the workpiece material T_{burn}.

In order to avoid thermal damage, we need to predict the allowable grinding power P_{burn} (W) from the kinematical and geometrical parameters, as well as from the thermal properties of the workpiece, namely, the thermal conductivity k_0 (W $\mathrm{m^{-1}K^{-1}}$) and the thermal diffusivity k ($\mathrm{m^2s^{-1}}$). According to Malkin's model for dry grinding [1], we have

$$P_{\text{burn}} = u_0 b v_f a + B\, b D^{1/4} v_f^{1/2} a^{1/4}, \tag{1}$$

where

$$B = \frac{k_0 T_{\text{burn}}}{1.13\sqrt{k}}, \tag{2}$$

and u_0 ($\mathrm{J\,m^{-3}}$) is the specific energy that is not converted into heat in the grinding process.

It is worth noting that in the literature [1, 3], $B = k_0 T_{\max}/(1.13\sqrt{k})$ instead of (2). We will clarify later on this kind of misunderstanding about $T_{\max}$ and T_{burn}. In any case, Malkin's model assumes a constant heat flux profile entering into the workpiece, a geometrical contact length [2, Eqn. 3-4]

$$2\ell = \sqrt{aD},$$

and an energy partition in the chip formation going to the workpiece of 55% [3]. Also, it assumes a high Peclet number

$$L = \frac{v_f \ell}{2k} \to \infty, \tag{3}$$

although, according to [3], (1) provides a good approximation for $L > 5$. Finally, Malkin's model assumes dry grinding. However, if film-boiling occurs, the convection due to the lubricant within the grinding zone can be neglected, which is the case of conventional grinding [4]. We will revisit these assumptions in Section 2 in order to enhance Malkin's model.

The adaptive control in the machining processes is widely use in industry [5]. It uses characteristic measurements of the process in order to lead the machining into a desired condition. Depending on the nature of the desired condition, we have adaptive control constraint (ACC) or adaptive control optimization (ACO). Acoustic Emission (AE) has been used as the characteristic measurement in the adaptive control of plunge grinding. For instance, in [6], AE signals were calibrated against major grinding parameters such as specific material removal rate, specific material removal, grinding wheel and workpiece diameters and wheel dressing conditions. Other adaptive controls uses real-time power measurement. For instance, a computerized ACO system for plunge grinding of steels is described in [7], where the removal rate is maximized, being subjected to constraints of surface finish and workpiece burn. Also in [8], an ACO is proposed in order to prevent thermal damage in shallow cut cylindrical and surface grinding, by controlling the peripheral velocity of the wheel and its feedrate. In the present study, we are going to measure also the machine power consumption. Our objective is to minimize the number of wheel dressings in order to increase the production. However, as the wheel wears out, the risk of thermal damage of the workpiece is greater. Therefore, our goal is precisely achieve both things simultaneously as much as possible, i.e. reduce wheel dressings avoiding thermal damage. For this purpose, we are going to control the cutting depth a, because is quite easy to do it and affects little the behavior of the wheel. Therefore, let us solve (1) for a defining

$$a = z^4, \tag{4}$$

$$q = \frac{B\,D^{1/4}}{u_0 v_f^{1/2}} > 0, \tag{5}$$

$$r = \frac{-P_{\text{burn}}}{u_0 b v_f} < 0, \tag{6}$$

so that (1) becomes a quartic equation without the quadratic and cubic terms (quartic-linear equation)

$$z^4 + q\,z + r = 0. \tag{7}$$

Equation (7) can be solved numerically, for instance, by using Newton-Raphson method [9, Sect. 3.4.2]. However, this method needs a starting iteration point close enough to the desired root in order to assure its convergence (see [9, Sect. 3.6.1] for pitfalls about numerical root finding methods). Nonetheless, quartic equations can be solved analytically, but in general, its solution is cumbersome to implement and compute (see [10] for different procedures for solving quartics). For instance, according to [11, Eqns. 9.6-7], the solution of the general quartic equation

$$x^4 + m_1 x^3 + m_2 x^2 + m_3 x + m_4 = 0, \tag{8}$$

is given in two stages. First, we have to compute a real root y_r of the *resolvent*, which is the following cubic equation

$$y^3 - m_2 y^2 + (m_1 m_3 - 4m_4)\, y + 4m_2 m_4 - m_3^2 - m_1^2 m_4 = 0. \quad (9)$$

Second, the four solutions of (8) are given by the four roots, $z_1^\pm$, $z_2^\pm$, of the quadratic equation

$$z^2 + \frac{m_1 \pm \sqrt{m_1^2 - 4m_2 + 4y_r}}{2} z + \frac{y_r \pm \sqrt{y_r^2 - 4m_4}}{2} = 0. \quad (10)$$

For the solution of the *resolvent* (9), the general solution of a cubic equation [11, Eqn. 9.3]

$$y^3 + \mu_1 y^2 + \mu_2 y + \mu_3 = 0, \quad (11)$$

is given by

$$y_1 = F + G - \frac{1}{3}\mu_1, \quad (12)$$

$$y_2 = -\frac{1}{2}(F+G) + \frac{\sqrt{3}}{2} i (F-G) - \frac{1}{3}\mu_1, \quad (13)$$

$$y_3 = -\frac{1}{2}(F+G) - \frac{\sqrt{3}}{2} i (F-G) - \frac{1}{3}\mu_1, \quad (14)$$

where

$$F = \sqrt[3]{V + \sqrt{U^3 + V^2}},$$

$$G = \sqrt[3]{V - \sqrt{U^3 + V^2}},$$

and

$$U = \frac{3\mu_2 - \mu_1^2}{9},$$

$$V = \frac{9\mu_1\mu_2 - 27\mu_3 - 2\mu_1^3}{54}.$$

The above solution for the general quartic equation is not very practical and somehow difficult to implement in the numerical control of the grinding machine, because, according to (9), we have first to seek a real solution y_r of (9), computing (12)-(14) with the proper coefficients. Moreover, since we are looking for a real and positive solution (remember $a = z^4 > 0$), we have to check all the solutions of (10) and neglect the complex and negative ones.

Fortunately, the quartic-linear equation given in (7), where $p > 0$ and $q < 0$, has got an unique positive root and there is a straightforward formula to find it. This statement is just one of the main results of the present paper. Moreover, if Malkin's model is enhanced in order to consider both an arbitrary heat flux profile and also the first order kinematic correction to the contact length, we will find a quartic equation like (7), so that the aforementioned formula remains applicable.

This paper is organized as follows. In Section 2 we will discuss the assumptions of Malkin's model in order to consider an arbitrary heat flux profile and the kinematic effect over the contact length. For the latter, in Appendix C we will provide a novel discussion about the exact and approximate kinematic contact length formulas. Section 3 describes the adaptive control of the cutting depth. Section 4 solves the quartic-linear equation stated in (7). For this purpose, we will use Appendix A for the solution of the cubic equation in terms of elementary functions, and Appendix B for the solution of Descartes about quartics. Section 5 shows some experimental results about the implementation of this method in the numerical control of a grinding machine. Finally, our conclusions are collected in Section 6.

	Symbol	**Meaning**	**SI units**
Wheel	D	Diameter	m
	R	Radius	m
	v_θ	Peripheral velocity	$\mathrm{m\,s^{-1}}$
Workpiece	k_0	Thermal conductivity	$\mathrm{W\ m^{-1}K^{-1}}$
	k	Thermal diffusivity	$\mathrm{m^2s^{-1}}$
	T	Temperature	K
	T_{burn}	Change of phase temperature	K
	$T_{\max}$	Maximum temperature	K
Grinding regime	a	Cutting depth	m
	b	Grinding width	m
	ℓ	Half contact length	m
	v_f	Feedrate	$\mathrm{m\,s^{-1}}$
	P_{burn}	Allowable grinding power	W
	P_{tot}	Total grinding power	W
	Q_{tot}	Total heat flux	$\mathrm{W\ m^{-2}}$
	Q	Average heat flux going to the workpiece	$\mathrm{W\ m^{-2}}$
	E_{tot}	Total energy	J
	E	Energy going to the workpiece	J
	u_{tot}	Total specific energy	$\mathrm{J\,m^{-3}}$
	u_{chip}	Specific energy going to the chip	$\mathrm{J\,m^{-3}}$
	u_0	Specific energy not converted into heat	$\mathrm{J\,m^{-3}}$

Table 1: Nomenclature for dimensional parameters.

Symbol	Meaning
$L = v_f \ell / (2k)$	Peclet number
$\mathcal{T} = \pi k_0 v_f T / (2kQ)$	Dimensionless temperature
$\mathcal{T}_{\max}$	Maximum dimensionless temperature
A_p	Constant of the heat flux profile
$\xi = a/R$	Dimensionless cutting depth
$\chi = v_f / v_\theta$	Speed ratio
$\varepsilon = E/E_{\text{tot}}$	Energy partition ratio
$s = u_0/u_{\text{chip}}$	Specific energy ratio
λ	Apex location of the triangular heat flux profile
f	Power reduction fraction
$f(x)$	Heat flux distribution

Table 2: Nomenclature for dimensionless parameters.

2 Malkin's model

2.1 Heat flux profile assumption

In surface grinding, the maximum dimensionless temperature $\mathcal{T}_{\max}$ for large Peclet numbers is given by [12]

$$\mathcal{T}_{\max} \approx A_p \sqrt{\pi L}, \quad L \to \infty, \tag{15}$$

where the constant A_p depends on the heat flux profile within the contact area between wheel and workpiece, and $\mathcal{T}$ is the dimensionless temperature, defined as

$$\mathcal{T} = \frac{\pi k_0 v_f}{2kQ} T, \tag{16}$$

being Q (W m^{-2}) the average heat flux entering into the workpiece. Table 3 shows the value of A_p for the most common heat flux profiles reported in the literature, namely, constant [13, 14], linear [14, 15], triangular [16, 17], and parabolic [18]. Notice that in the triangular profile, λ is a dimensionless parameter that denotes the location of the apex (see Fig. 2). In [12] there is a discussion about the accuracy of (15), depending on L. It is concluded that for a constant heat flux profile, the error is about 5% for $L > 5$; and for a linear, parabolic or triangular heat flux profile, the error is $< 1\%$ for $L > 5$.

Heat flux profile	A_p
Constant	2
Linear	$\frac{4}{3}\sqrt{2}$
Triangular	$\frac{8}{3}\sqrt{\frac{2}{3-\lambda}}$
Parabolic	$\frac{2}{5}\sqrt{6\left(3+\sqrt{3}\right)}$

Table 3: Dimensionless heat flux profile factor for maximum temperature.

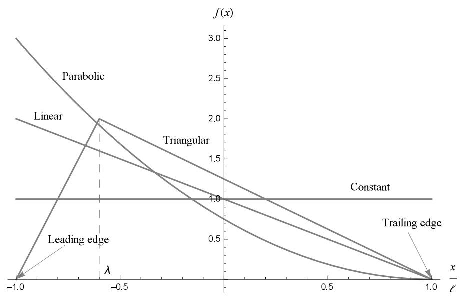


Fig. 2 shows the dimensionless heat flux distribution $f(x)$ over the contact length, normalized to unity, that is

$$\frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx = 1. \tag{17}$$

It is worth noting that since (15) is based on a previous result given in [19], similar calculations can be performed for any other kind of heat flux profile $f(x)$, provided that $f(x)$ is a piecewise analytic function.

2.2 Contact length assumption

The widely used formula for the geometrical contact length

$$2\ell = \sqrt{aD} \tag{18}$$

is in fact an approximation of the exact formula [2, Eqn. 3-2]

$$2\ell = R\cos^{-1}\left(1-\frac{a}{R}\right), \tag{19}$$

where R is just the radius of the wheel

$$D = 2R. \tag{20}$$

Geometrically speaking, (19) calculates the arc length of the wheel in contact to the workpiece, meanwhile (18) is just the chord of that arc. In surface grinding, where the depth of cut a is much smaller than the wheel radius, that is

$$\xi = \frac{a}{R} \ll 1, \tag{21}$$

the substitution of the exact formula (19) by the approximate one (18) is justified due to the expansion [20, Eqn. 35:6:3]

$$\cos^{-1}(1-\xi) = \sqrt{2\xi}\left(1 + \frac{\xi}{12} + \frac{3\xi^2}{160} + \frac{5\xi^3}{896} + \cdots\right), \quad 0 \le \xi \le 2,$$

so that, expanding (19) up to first order, we get (18).

Nonetheless, the workpiece is moving with respect to the wheel, thus the path traveled by a grain on the wheel surface in contact with the workpiece is different from the geometrical contact length given in (18) or (19). Therefore, there is a kinematical correction to the static geometrical contact length. In surface grinding, this correction is usually given by the following formula [24]

$$2\ell = (1+\chi)\sqrt{aD}, \tag{22}$$

where the speed ratio χ is a dimensionless parameter given by

$$\chi = \frac{v_f}{v_\theta}. \tag{23}$$

However, equation (22) is also an approximation and the kinematic contact length can be calculated exactly as

$$2\ell = D(1+\chi)\ \mathrm{E}\left(\frac{4\chi}{(1+\chi)^2}, \frac{\cos^{-1}(1-\xi)}{2}\right), \tag{24}$$

where $\mathrm{E}(k,\phi)$ is the *elliptic integral of the second kind* [20, Eqn. 62:3:2]. Surprisingly, (24) seems not to be reported in the literature, so a formal derivation is given in Appendix C. In surface grinding, we can justify the usage of the approximate formula (22), since, according to (19) and (21), we have

$$\cos^{-1}(1-\xi) \approx 0,$$

thus, applying to (24) the following limiting approximation [20, Eqn. 62:9:2]

$$\mathrm{E}(k,\phi) \approx \phi - \frac{k}{6}\phi^3 + \cdots, \quad \phi \approx 0,$$

we get

$$2\ell \approx R(1+\chi)\cos^{-1}(1-\xi). \tag{25}$$

Now, recalling that (19) is approximated by (18), then (25) becomes (22), as we wanted to prove.

2.3 Malkin's model revisited

Taking into account (3) and (16), we can rewrite (15) in dimensional form as

$$T_{\max} = \frac{A_p}{\sqrt{\pi}} \frac{Q\sqrt{2\ell}\sqrt{k}}{k_0\sqrt{v_f}}, \tag{26}$$

where, remember that (26) is an approximation that is asymptotically satisfied for $L \to \infty$. Defining Q_{tot} as the total power P_{tot} consumed in the process per unit surface in contact to the wheel, that is

$$Q_{\text{tot}} = \frac{P_{\text{tot}}}{2\ell b}, \tag{27}$$

and u_{tot} ($\mathrm{J\,m^{-3}}$) as the total specific energy of the process (energy consumed in the process E_{tot} per unit volume of material removed)

$$u_{\text{tot}} = \frac{dE_{\text{tot}}}{dV} = \frac{dE_{\text{tot}}/dt}{dV/dt} = \frac{P_{\text{tot}}}{abv_f}, \tag{28}$$

we have that

$$Q_{\text{tot}} = \frac{av_f u_{\text{tot}}}{2\ell}. \tag{29}$$

Now, let us set the dimensionless parameter ε as the ratio of the energy entering into the workpiece E with respect to the total energy consumed in the process E_{tot}, thus

$$\varepsilon = \frac{E}{E_{\text{tot}}} = \frac{Q}{Q_{\text{tot}}} = \frac{u}{u_{\text{tot}}}, \tag{30}$$

where u is the fraction of specific energy going to the workpiece. Therefore, from (29) and (30), we obtain

$$Q = \frac{Q_{\text{tot}}}{u_{\text{tot}}} u = \frac{av_f}{2\ell} u. \tag{31}$$

According to Malkin's model, all the specific energy u_{tot}, except a fraction s of the specific energy invested to generate chip u_{chip}, goes to the workpiece, so

$$u = u_{\text{tot}} - s\, u_{\text{chip}}, \tag{32}$$

where, according to [3] $s \approx 0.45$. Therefore, substituting (32) in (31) and setting $u_0 = s\, u_{\text{chip}}$, we get

$$Q = \frac{av_f}{2\ell} \left(u_{\text{tot}} - u_0 \right). \tag{33}$$

Now, substituting (33) in (26), we have

$$T_{\max} = \frac{A_p}{\sqrt{\pi}} \frac{a\sqrt{kv_f}}{k_0\sqrt{2\ell}} \left(u_{\text{tot}} - u_0 \right), \tag{34}$$

and substituting in (34) the kinematic contact length approximation given in (22), we obtain, solving for u_{tot},

$$u_{\text{tot}} = u_0 + \frac{\sqrt{\pi}}{A_p} \frac{k_0 T_{\max}\sqrt{1+\chi}}{\sqrt{kv_f}} D^{1/4} a^{-3/4}. \tag{35}$$

Multiplying (35) by the volumetric removal rate

$$\frac{dV}{dt} = abv_f,$$

and taking into account (28), we finally get

$$P_{\text{tot}} = u_0 b v_f a + \frac{\sqrt{\pi}}{A_p} \frac{b k_0 T_{\max} \sqrt{1+\chi}}{\sqrt{k}} \sqrt{v_f} D^{1/4} a^{1/4}. \tag{36}$$

When the maximum temperature reached in the workpiece equals the change of phase temperature of the material, i.e. $T_{\max} = T_{\text{burn}}$, then the total power consumed in the grinding process reaches the maximum allowable power, i.e. $P_{\text{tot}} = P_{\text{burn}}$, namely, when the workpiece starts getting burnt. Therefore, in this case, we have

$$P_{\text{burn}} = u_0 b v_f a + B_p b v_f^{1/2} D^{1/4} a^{1/4}, \tag{37}$$

where

$$B_p = \frac{\sqrt{\pi}}{A_p} \frac{k_0 T_{\text{burn}} \sqrt{1+\chi}}{\sqrt{k}}. \tag{38}$$

Notice that when we have a constant heat flux profile (i.e. $A_p = 2$) and there is not kinematical correction (i.e. $\chi = 0$), then $B_p = B$, and (37) becomes (1). Therefore, (37) provides a generalization of Malkin's model, although the quartic-linear equation we need to solve in (7) remains the same, being the only change to perform the replacement of B by B_p in (5).

As aforementioned, it is worth noting that there is a kind of misunderstanding about $T_{\max}$ and T_{burn} in the literature, (see, for instance [1, 3]), because it seems that they are conceptually equivalent. However, $T_{\max}$ refers to the maximum temperature in the workpiece for given grinding conditions, and T_{burn} refers to the maximum temperature allowable by the workpiece material, so that the latter is not thermally damaged.

3 The adaptive control of the grinding process

In real conditions, the power consumed P_{tot} in the grinding process rises as the wheel wears out, because the wear flat area of the abrasive grains is augmented, so that the energy per unit time and per unit area Q entering into the workpiece is increased as well. Therefore, P_{tot} eventually reaches or even exceeds P_{burn}, and thermal damage occurs. In order to avoid this, we can change the grinding conditions, so that P_{tot} is reduced. For instance, according to (36), we can lower v_f or a, but the variation of a affects less to the behavior of the wheel (its hardness), so the latter is preferred. Notice that if the depth of cut a is reduced, according to (36) and (37), both P_{tot} and P_{burn} will be lowered. However, P_{tot} is reduced more than P_{burn} since T_{burn} is a constant independent of a, and $T_{\max}$ depends on a, according to (34). Nonetheless, P_{tot} is increased in time again, so that we have to reduce again the depth of cut a. Doing so periodically before P_{burn} is reached, we will assure the quality of the workpiece and we will increase the production, since we will reduce the number of wheel dressings.

Mathematically speaking, the adaptive control explained above involves the following steps:

- Setting the geometrical and kinematical grinding parameters (i.e. b, D, v_f and v_θ) and knowing the thermal properties of the workpiece (i.e. k, k_0 and T_{burn}), as well as the heat flux profile (i.e. A_p) and the specific energy not converted into heat (i.e. u_0); for an initial depth of cut a_0, we can calculate an initial allowable grinding power $P_{\text{burn},0}$ by using (37).

- Monitoring in time the actual power consumed in the grinding process P_{tot}, when $P_{\text{tot}} \geq P_{\text{burn},0}$, we will recalculate the depth of cut a_1, so that now, the allowable grinding power is reduced a fixed fraction f of its initial quantity: $P_{\text{burn},1} = (1 - f) P_{\text{burn},0}$. Notice that in order to calculate a_1, we need to insert $P_{\text{burn},1}$ in (37) and then solve for the cutting depth, i.e. to solve the quartic-linear equation given in (4)-(7).

- In general, when $P_{\text{tot}} \geq P_{\text{burn},j}$ $(j = 0, 1, 2 \ldots)$ we will reset the allowable grinding power as $P_{\text{burn},j+1} = (1 - (j + 1)\, f) P_{\text{burn},0}$ and then we will recalculate the depth of cut a_{j+1} similarly as aforementioned.

4 The quartic-linear equation

Let us solve now the quartic equation presented in the Introduction, that is, a fourth degree equation without the quadratic and cubic terms. According to (5)-(6) and (7), this equation reads as

$$\begin{gathered} z^4 + q\, z + r = 0, \\ q > 0, r < 0. \end{gathered} \tag{39}$$

Theorem 1 *The quartic-linear equation given in (39) has got an unique positive real root*

$$z = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right), \tag{40}$$

where

$$\alpha = \sqrt{-2\sqrt{m}\sinh\left(\frac{1}{3}\sinh^{-1}\left(n\, m^{-3/2}\right)\right)} > 0, \tag{41}$$

$$\beta = \frac{2\alpha r}{\alpha^3 + q} < 0, \tag{42}$$

$$m = -\frac{4}{3} r > 0, \tag{43}$$

$$n = -\frac{q^2}{2} < 0. \tag{44}$$

Proof. *Following the method described in the Appendix B, since (39) has not cubic term, it can be factored as*

$$\left(z^2 + \alpha\, z + \beta\right)\left(z^2 - \alpha\, z + \gamma\right) = 0.$$

In order to calculate the coefficients α, β and γ, we have to solve first the resolvent given in (82), but taking $p = 0$, because the coefficient of the quadratic term in (39) is null. Thus, we have

$$\alpha^6 - 4r\alpha^2 - q^2 = 0. \tag{45}$$

Notice that (45) is a cubic equation in α^2 without the quadratic term, in which, according to (39), the linear coefficient is positive, $-4r > 0$. Therefore, we can apply the solution of Case I given in Appendix A. Since we are looking for real solutions, we just take the solution given in (73)-(74), that is

$$\alpha^2 = -2\sqrt{m}\sinh\left(\frac{1}{3}\sinh^{-1}\left(n\, m^{-3/2}\right)\right), \tag{46}$$

where, comparing (45) with (59), we have

$$m = -\frac{4}{3}r > 0, \tag{47}$$

$$n = -\frac{q^2}{2} < 0. \tag{48}$$

Since the $\sinh$ *and* $\sinh^{-1}$ *functions are odd functions [11, Eqns. 8.14, 8.64], taking into account (47)-(48) over (46), we can conclude easily that*

$$\alpha > 0. \tag{49}$$

Once α is known, we calculate the coefficients β and γ particularizing (85) and (86) with $p = 0$, so that

$$\gamma = \frac{\alpha^3 + q}{2\alpha} > 0, \tag{50}$$

$$\beta = \frac{2\alpha r}{\alpha^3 + q} < 0. \tag{51}$$

where the signs of (50) and (51) are easily derived from (49) and (39). Now, according to the signs obtained for α, β and γ, the domain of the solutions of the quartic-linear equation (39) are easily derived from (87)-(88). For instance, since $\beta < 0$, then

$$\alpha^2 - 4\beta > \alpha^2 > 0. \tag{52}$$

Applying now the square root to both sides of (52), knowing that the square root is an increasing function for positive arguments, then the inequality holds, thus

$$\sqrt{\alpha^2 - 4\beta} > \alpha, \tag{53}$$

so, according to the + solution given in (87),

$$z_1^+ = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right) > 0.$$

Now, from (49) and (53), we have

$$\sqrt{\alpha^2 - 4\beta} > 0,$$

thus, according to the − solution given in (87),

$$z_1^- = \frac{1}{2}\left(-\alpha - \sqrt{\alpha^2 - 4\beta}\right) < 0.$$

Finally, taking into account (50) and (39), we have

$$\alpha^2 - 4\gamma = -\alpha^2 - q < 0,$$

thus, the solutions given in (88) are not real-valued

$$z_2^\pm = \frac{1}{2}\left(\alpha \pm \sqrt{\alpha^2 - 4\gamma}\right) \notin \mathbb{R}.$$

Therefore, the unique positive real root of (39) is given by z_1^+, as stated in (40).
■

Since (40) is the unique positive root of (39), recalling the change of variables performed in (4), the depth of cut a is calculated as

$$a = \frac{1}{16}\left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right)^4. \tag{54}$$

where α and β are given by (41) and (42) respectively.

5 Experimental results

Tables 4 and 5 collect the input parameters used in the experimental tests, according to the enhanced model of Malkin discussed in Section 2. It is worth noting that it has been supposed a linear heat flux profile, taking $A_p = \frac{4}{3}\sqrt{2}$ (see Table 3), since it is the one that fits better to the thermography experimental values in surface dry grinding [17]. Also, the value of u_0 is the one corresponding to steels [2, p. 166], since the workpiece material is steel 1.2842 (F5229).

Once the input parameters are set, we can calculate the initial allowable power (37)

$$P_{\text{burn},0} = u_0 b v_f a_0 + B_p b v_f^{1/2} D^{1/4} a_0^{1/4}. \tag{55}$$

During the adaptive control iterations, the successive allowable powers are calculated as

$$P_{\text{burn},j+1} = (1 - (j+1)\ f)\, P_{\text{burn},0}, \quad j = 0, 1, 2, \ldots \tag{56}$$

Parameter	Magnitude	SI units
v_θ	35	$\mathrm{m\,s^{-1}}$
D	0.3847	m
b	8.6×10^{-3}	m
k_0	52.5	$\mathrm{W\ m^{-1}K^{-1}}$
k	1.4535×10^{-5}	$\mathrm{m^2s^{-1}}$
A_p	$\frac{4}{3}\sqrt{2}$	—
T_{burn}	700	K
u_0	6.2×10^9	$\mathrm{J\,m^{-3}}$

Table 4: Input parameters.

	Wheel Reference	v_f $[\mathrm{m\,s^{-1}}]$	a_0 [m]
Test 1	2AMBA46G12 V81 P24P	0.1667	5.40×10^{-5}
Test 2	5MBA46G12 V489 P24P	0.1667	4.10×10^{-5}
Test 3	MA46G12 V489 P24P	0.25	3.80×10^{-5}
Test 4	CBL36.2H10 V489 P24P	0.25	2.80×10^{-5}

Table 5: Grinding conditions for the tests.

where we reduce a 10% of $P_{\text{burn},0}$ in each iteration, i.e. $f = 0.1$. Now, according (5)-(6) and theorem 1, we calculate the following parameters:

$$
\begin{aligned}
q &= \frac{B_p\, D^{1/4}}{u_0 v_f^{1/2}}, \quad r = \frac{-P_{\text{burn},j+1}}{u_0 b v_f}, \\
m &= -\frac{4}{3} r, \quad n = -\frac{q^2}{2}, \\
\alpha &= \sqrt{-2\sqrt{m}\sinh\left(\frac{1}{3}\sinh^{-1}\left(n\, m^{-3/2}\right)\right)}, \\
\beta &= \frac{2\alpha r}{\alpha^3 + q},
\end{aligned}
$$

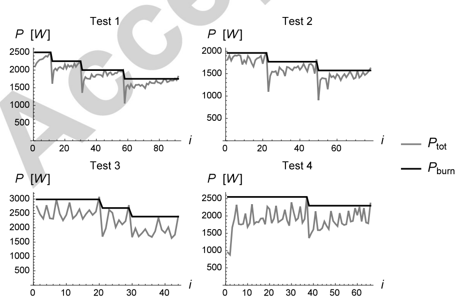
so that, we can calculate straightforwardly the depth of cut a_{j+1} as

$$
a_{j+1} = \frac{1}{16}\left(-\alpha + \sqrt{\alpha^2 - 4\beta}\right)^4, \quad j = 0, 1, 2, \ldots \tag{57}
$$

Table 6 shows the iterations in the adaptive control of the grinding process, by using (55)-(57).

	Test 1		**Test 2**	
j	$P_{\text{burn},j}$ [W]	a_j [m]	$P_{\text{burn},j}$ [W]	a_j [m]
0	2501.8	5.40×10^{-5}	1963.6	4.10×10^{-5}
1	2251.8	4.08×10^{-5}	1766.9	3.05×10^{-5}
2	2000.5	2.92×10^{-5}	1572.0	2.14×10^{-5}
3	1752.2	1.95×10^{-5}		
j	**Test 3**		**Test 4**	
0	2988.9	3.80×10^{-5}	2250.5	2.80×10^{-5}
1	2688.4	2.86×10^{-5}	2297.0	2.07×10^{-5}
2	2389.4	2.03×10^{-5}		

Table 6: Allowable grinding power and cutting depth for each iteration.



In Fig. 3 is shown the time evolution of the power consumption P_{tot} and the allowable power P_{burn} during the adaptive control tests. The grinding conditions of the different tests are given in Table 5. In the abscissas, the number of depths i of the grinding wheel over the workpiece surface is represented. Notice that when we change the cutting depth in each iteration, P_{tot} lowers more than P_{burn} as aforementioned in Section 3. Also, we can see that the adaptive control extends the operation of the wheel in a factor between 2 to 6 times, without needing dressing the wheel and preserving the surface integrity of the workpiece.

6 Conclusions

We have derived a closed analytical formula in terms of elementary functions to solve the unique positive real root of a quartic equation without the cubic and quadratic terms (quartic-linear equation). It turns out that this formula is quite useful for the adaptive control of the depth of cut in surface dry grinding in order to avoid thermal damage, because is very easy to implement in the numerical control of the grinding process and also it is extremely fast to compute. Unlike other authors, we have chosen to control the cutting depth instead of the feedrate, because the variation of the depth of cut affects less the behavior of the wheel (its hardness).

From the point of view of production, the smaller is the number of wheel dressings, the better. However, as the wheel wears out, the risk of thermal damage of the workpiece is greater. Therefore, the goal of the adaptive control presented here is precisely to provide both things simultaneously as much as possible. In the adaptive tests presented in this paper, we have increased very significantly the number of passes of the wheel over the workpiece, for different types of grinding wheels, feedrates and initial cutting depths.

In addition, Malkin's model, in which is based the prediction of thermal damage in surface dry grinding, has been enhanced in order to consider different heat flux profiles in the grinding zone, and the kinematical correction to the geometrical contact length. The derived formula for the quartic-linear equation is still being applicable for the latter case.

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Conflict of Interests The authors declare that there is no conflict of interests regarding the publication of this paper.

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Appendix

A Cubic equation

Any cubic equation with real coefficients can be expressed as

$$z^3 + a\,z^2 + b\,z + c = 0, \quad a, b, c \in \mathbb{R}. \tag{58}$$

Performing in (58) the change of variables

$$z = x - \frac{a}{3},$$

we obtain a cubic equation without the quadratic term, which is know as *depressed cubic equation*. This equation reads as

$$x^3 \pm 3m\,x + 2n = 0, \tag{59}$$

where $m > 0$ and

$$\begin{aligned} \pm 3m &= b - \frac{a^2}{3}, \\ 2n &= \frac{2a^3}{27} - \frac{ab}{3} + c. \end{aligned} \tag{60}$$

Performing now in (59) the change of variables

$$x = u + v, \tag{61}$$

we arrive at

$$u^3 + v^3 + 3\,(u + v)\,(uv \pm m) + 2n = 0. \tag{62}$$

Equation (62) holds if u and v satisfy

$$u^3 + v^3 + 2n = 0, \tag{63}$$

$$uv \pm m = 0. \tag{64}$$

Solving (64), we obtain

$$v = \mp \frac{m}{u}, \tag{65}$$

and substituting (65) in (63), we arrive at

$$u^6 + 2n\,u^3 \mp m^3 = 0. \tag{66}$$

Equation (66) is a quadratic equation in u^3 that can be solved easily, obtaining

$$u^3 = -n \pm \sqrt{n^2 \pm m^3}, \tag{67}$$

so that, substituting (67) in (65), we have

$$v^3 = \mp \frac{m^3}{u^3} = -n \mp \sqrt{n^2 \pm m^3}. \tag{68}$$

Notice that the solutions for u and v given in (67) and (68) are interchangeable, so that we can choose the $+$ sign before the radical in (67) and the sign $-$ in (68), without loss of generality. Finding the cubic roots in (67) and (68), we have the following set of solutions:

$$\begin{aligned} u_1 &= \left(-n + \sqrt{n^2 \pm m^3}\right)^{1/3}, \\ u_2^{\pm} &= u_1 \exp\left(\pm \frac{2\pi i}{3}\right) = \frac{1}{2}\left(-1 \pm \sqrt{3}i\right) u_1, \end{aligned}$$

and

$$\begin{aligned} v_1 &= \left(-n - \sqrt{n^2 \pm m^3}\right)^{1/3}, \\ v_2^{\pm} &= v_1 \exp\left(\pm \frac{2\pi i}{3}\right) = \frac{1}{2}\left(-1 \pm \sqrt{3}i\right) v_1. \end{aligned}$$

Now, according to (61), we finally arrive at the three solutions of the cubic equation in the classical form given by Cardan [21]

$$\begin{aligned} x_1 &= u_1 + v_1 \\ &= \left(-n + \sqrt{n^2 \pm m^3}\right)^{1/3} + \left(-n - \sqrt{n^2 \pm m^3}\right)^{1/3}, & (69) \\ x_2^{\pm} &= u_2^{\pm} + v_2^{\mp} \\ &= -\frac{1}{2}(u_1 + v_1) \pm \frac{\sqrt{3}i}{2}(u_1 - v_1). & (70) \end{aligned}$$

A.1 Solution in terms of elementary functions

Cardan's classical solution can be expressed in a much convenient way [22] by using the following identities:

$$\pm x + \sqrt{x^2 + a^2} = a \exp\left(\pm \sinh^{-1}\left(\frac{x}{a}\right)\right), \tag{71}$$

$$x \pm \sqrt{x^2 - a^2} = \begin{cases} a \exp\left(\pm \cosh^{-1}\left(\frac{x}{a}\right)\right), & x^2 - a^2 \geq 0, \\ a \exp\left(\pm i \cos^{-1}\left(\frac{x}{a}\right)\right), & x^2 - a^2 \leq 0. \end{cases} \tag{72}$$

$$a \in \mathbb{R}.$$

Formula (71) comes directly from the definition of the $\sinh^{-1}$ function [11, Eqn. 8.55]. Formula (72) considers respectively the positive and negative branches of the cosh function [11, Eqn. 8.56], where in the negative branch, it has taken into account the property $\cos^{-1} u = \pm i \cosh^{-1} u$ [11, Eqn. 8.94].

Therefore, applying (71)-(72) to the solutions given in (69)-(70), we can differentiate three cases.

Case I $+$ sign in (59) (recalling that $m > 0$). One real root and two conjugate complex roots.

$$\begin{aligned} x_1 &= -2\sqrt{m}\sinh\theta_1, \\ x_2^{\pm} &= \sqrt{m}\left(\sinh\theta_1 \pm i\sqrt{3}\cosh\theta_1\right), \end{aligned} \tag{73}$$

where

$$\theta_1 = \frac{1}{3}\sinh^{-1}\left(n\, m^{-3/2}\right). \tag{74}$$

Case II $-$ sign in (59) and $n^2 - m^3 > 0$. One real root and two conjugate complex roots.

$$\begin{aligned} x_1 &= -2\sqrt{m}\cosh\theta_2, \\ x_2^{\pm} &= \sqrt{m}\left(\cosh\theta_2 \pm i\sqrt{3}\sinh\theta_2\right), \end{aligned}$$

where

$$\theta_2 = \frac{1}{3}\cosh^{-1}\left(n\, m^{-3/2}\right).$$

Case III $-$ sign in (59) and $n^2 - m^3 \leq 0$. Three real roots (trigonometric solution).

$$\begin{aligned} x_1 &= -2\sqrt{m}\cos\theta_3, \\ x_2^{\pm} &= \sqrt{m}\left(\cos\theta_3 \pm \sqrt{3}\sin\theta_3\right), \end{aligned}$$

where

$$\theta_3 = \frac{1}{3}\cos^{-1}\left(n\, m^{-3/2}\right).$$

This solution can be given in a compact way as

$$x_k = -2\sqrt{m}\cos\left(\frac{\theta_3 + 2\pi k}{3}\right), \quad k = 0, 1, 2.$$

B Quartic equation

Any quartic equation with real coefficients can be expressed as

$$z^4 + a\, z^3 + b\, z^2 + c\, z + d = 0, \quad a, b, c, d \in \mathbb{R}. \tag{75}$$

Performing the following change of variables in (75)

$$z = x - \frac{a}{4},$$

results in a quartic equation without the cubic term

$$z^4 + p\,z^2 + q\,z + r = 0, \tag{76}$$

where

$$\begin{aligned} p &= -\frac{3}{8}a^2 + b, \\ q &= \frac{a^3}{8} - \frac{ab}{2} + c, \\ r &= -\frac{3a^4}{256} + \frac{a^2 b}{16} - \frac{ac}{4} + d. \end{aligned}$$

The idea of Descartes [23] consists in factoring the quartic polynomial given in (76) into two quadratic polynomials as follows

$$\left(z^2 + \alpha\,z + \beta\right)\left(z^2 - \alpha\,z + \gamma\right) = 0, \tag{77}$$

where, identifying coefficients from (76) and (77), we have

$$p = \gamma + \beta - \alpha^2, \tag{78}$$

$$q = \alpha\left(\gamma - \beta\right), \tag{79}$$

$$r = \gamma\beta. \tag{80}$$

Notice that the factorization given in (77) has been possible because (76) has not cubic term. In order to solve for α, from (78) and (79), we can do the following

$$\begin{aligned} \left(p + \alpha^2\right)^2 - \left(\frac{q}{\alpha}\right)^2 &= (\gamma + \beta)^2 - (\gamma - \beta)^2 \\ &= 4\gamma\beta = 4r, \end{aligned} \tag{81}$$

where we have taken into account (80). Multiplying (81) by α^2 and arranging terms, we arrive at

$$\alpha^6 + 2p\alpha^4 + \left(p^2 - 4r\right)\alpha^2 - q^2 = 0. \tag{82}$$

Equation (82) is known as the *resolvent* and it is a cubic equation in α^2, thus it can be solved by using the method described in Section A. Once α is known, we can solve for γ from (78) and (79)

$$p + \alpha^2 = \gamma + \beta, \tag{83}$$

$$\frac{q}{\alpha} = \gamma - \beta, \tag{84}$$

so adding up (83) and (84), we obtain

$$\gamma = \frac{1}{2}\left(p + \alpha^2 + \frac{q}{\alpha}\right). \tag{85}$$

Once γ is known, from (80), β is directly solved as

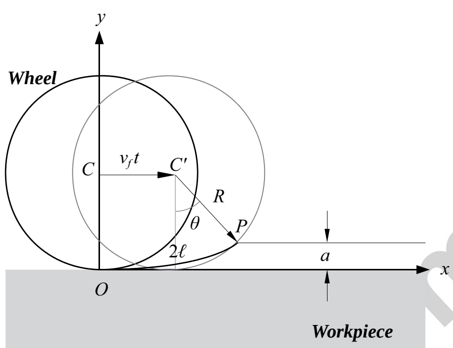
$$\beta = \frac{r}{\gamma}. \tag{86}$$

Known α, β and γ, we can solve easily (77), obtaining finally the four solutions of the quartic equation as

$$z_1^{\pm} = \frac{1}{2}\left(-\alpha \pm \sqrt{\alpha^2 - 4\beta}\right), \tag{87}$$

$$z_2^{\pm} = \frac{1}{2}\left(\alpha \pm \sqrt{\alpha^2 - 4\gamma}\right). \tag{88}$$

C Kinematic contact length



In Fig. 4, the Cartesian coordinate system OXY is fixed to the workpiece. Initially (at $t = 0$) the center of the wheel is at C and in time t a grain travels from O to P. The trajectory described by this grain is the composition of two movements: the shift of the center of the wheel from C to C' and the rotation of the wheel around its axis an angle θ.

If the wheel rotates at a constant angular velocity ω_θ, we have

$$\theta = \omega_\theta t, \tag{89}$$

The Cartesian coordinates of point P are given by the following parametric equations:

$$\begin{aligned} x(t) &= R \sin \omega_\theta t + v_f t, \\ y(t) &= R(1 - \cos \omega_\theta t), \end{aligned} \tag{90}$$

thus

$$\begin{aligned} x'(t) &= v_\theta \cos \omega_\theta t + v_f, \\ y'(t) &= v_\theta \sin \omega_\theta t. \end{aligned} \tag{91}$$

Since v_θ is the peripheral velocity of the wheel, we have

$$v_\theta = R\,\omega_\theta. \tag{92}$$

If at time t point P is located at a distance apart to the abscissa equal to the cutting depth (i.e. $y\,(t) = a$), then, according to (90), we have

$$a = R\,(1 - \cos\omega_\theta t)\,, \tag{93}$$

so, taking into account the dimensionless parameter given in (21), $\xi = a/R$, we have

$$\omega_\theta t = \cos^{-1}(1 - \xi)\,. \tag{94}$$

The kinematic contact length 2ℓ is then the arc length of the grain trajectory from O to P, thus from (91), and performing the change of variables $u = \omega_\theta \tau$, we have

$$\begin{aligned} 2\ell &= \int_0^t \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2}\,d\tau \\ &= \int_0^t \sqrt{v_\theta^2 + v_f^2 + 2v_\theta v_f \cos\omega_\theta\tau}\,d\tau \\ &= \frac{1}{\omega_\theta}\int_0^{\omega_\theta t} \sqrt{v_\theta^2 + v_f^2 + 2v_\theta v_f \cos u}\,du. \end{aligned} \tag{95}$$

Taking into account (92) and the dimensionless parameter defined in (23) $\chi = v_f/v_\theta$, we can rewrite (95) as

$$2\ell = R\int_0^{\omega_\theta t} \sqrt{1 + \chi^2 + 2\chi\cos u}\;du. \tag{96}$$

The integral given in (96) can be calculated by using the following formula reported in the literature [25, Eqn. 1.5.20(1)], [26, Eqn. 2.576.1], [27, Eqn. 289.01]

$$\int_0^\beta \sqrt{a + b\cos t}\,dt \stackrel{?}{=} 2\sqrt{a+b}\;\;\mathrm{E}\left(\sqrt{\frac{2b}{a+b}}, \frac{\beta}{2}\right), \qquad a > b > 0, \quad 0 \le \beta \le \pi. \tag{97}$$

where

$$\mathrm{E}\,(k, \phi) = \int_0^\phi \sqrt{1 - k\sin^2\theta}\;d\theta, \tag{98}$$

is the *elliptic integral of the second kind* [20, Eqn. 62:3:2]. However, performing the derivative on the right hand side of (97), we do not get the integrand of (97). Therefore, let us rewrite (96) as follows

$$2\ell = R\int_0^{\omega_\theta t} \sqrt{(1+\chi)^2 - 2\chi\,(1 - \cos u)}\;du,$$

and then let us apply the half angle formula for the sine function $2\sin^2(u/2) = 1 - \cos u$ and perform the change of variables $w = u/2$, thus

$$\begin{aligned} 2\ell &= R\int_0^{\omega_\theta t}\sqrt{(1+\chi)^2 - 4\chi\sin^2\frac{u}{2}}\,du \\ &= 2R(1+\chi)\int_0^{\omega_\theta t/2}\sqrt{1-\frac{4\chi}{(1+\chi)^2}\sin^2 w}\,dw. \end{aligned}$$

Taking into account the definition of the *elliptic integral of the second kind* given in (98) and also (94) and (20), we finally get

$$\begin{aligned} 2\ell &= 2R(1+\chi)\ \mathrm{E}\left(\frac{4\chi}{(1+\chi)^2}, \frac{\omega_\theta t}{2}\right) \\ &= D(1+\chi)\ \mathrm{E}\left(\frac{4\chi}{(1+\chi)^2}, \frac{\cos^{-1}(1-\xi)}{2}\right). \end{aligned} \tag{99}$$