Origin of the Kerker phenomena

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We provide an insight into the origin of the phenomena reported 40 years ago by Kerker *et al.* [J. Opt. Soc. Am. **73**, 765 (1983)]. We show that the impedance and refractive index matching conditions, discussed in Secs. II and IV of the seminal paper, are intertwined and both lead to the conservation of a Casimir invariant. We derive our results starting from the theory of representations of the Poincaré group, as it is a theory on which one of the most elemental descriptions of electromagnetic waves is based. We show that fundamental features of electromagnetic waves in continuous material environments can be derived by applying the symmetry-breaking principle. In particular, we identify the Casimir invariants of the $P_{3,1}$ subgroup as the magnitudes that describe the nature of monochromatic electromagnetic waves propagating in matter. Finally, we show that the emergence of the Kerker phenomena is associated with the conservation of such Casimir invariants in piecewise homogeneous media.

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I. INTRODUCTION

The study of the so-called Kerker conditions [1,2] has gathered an important amount of scientific efforts in the last decade [3–19]. The phrase first Kerker condition refers to scatterers that behave as those described in Sec. II of Kerker's seminal paper [1]. On the other hand, the phrase second Kerker condition has been usually employed to make reference to one type of scatterers discussed in Sec. III. The key characteristics of the scatterers fulfilling either of these two conditions are related to the preservation of helicity and the directionality of the emission. In particular, samples fulfilling Kerker's first condition are characterized for producing a scattered field with the same helicity as the illuminating excitation. As preservation of helicity is related to the restoration of duality symmetry [20], scatterers fulfilling the first Kerker condition are commonly called dual [21]. Scatterers fulfilling Kerker's second condition, however, are characterized for producing a scattered field of the opposite helicity. Consequently, they have usually been denoted as antidual. In addition, in the specific case of rotationally symmetric scatterers, the directionality of the emitted field is directly correlated with its helicity. For example, cylindrical dual scatterers do not emit in the backward direction, whereas cylindrical antidual scatterers do not emit in the forward direction [22,23].

The features described above have frequently placed the discussion about the two Kerker conditions at the same level. However, we should bear in mind that scatterers fulfilling the first or the second Kerker conditions have contradictory behaviors in many important aspects. For instance, dual scatterers can generally be defined in terms of material constants, i.e., they are achieved whenever the impedance of the sample and the surrounding medium is matched. Antidual scatterers, however, have only been identified in terms of multipolar scattering coefficients [22,24]. Moreover, whereas nonmagnetic scatterers that preserve helicity have been experimentally reported several times in the literature, antidual scatterers have fundamental problems regarding conservation of energy. These difficulties impelled the search for alternative definitions of the second Kerker condition that respected the optical theorem. The efforts resulted in the notion of the generalized second Kerker condition, which minimizes the forward scattered emission for a fixed scattering cross section [3]. Even if the generalized second Kerker condition was shown to be compatible with the conservation of electromagnetic energy, it is still a definition based on the Mie scattering coefficients of a sphere. Recently, it has been shown that antidual scatterers of any size and form have, by construction, a null extinction cross section [25,26].

The aforementioned differences in the description of dual and antidual scatterers led us to the conclusion that the current description of the two Kerker conditions is not satisfactory. Even if they have been usually discussed on the same footing, it is clear that the nature of these two effects

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is dissimilar in many important aspects. Here, we propose an alternative approach to understand the Kerker phenomena with the introduction of the resonant helicity mixing condition. This phenomenon is partially studied in Sec. IV of Kerker's original paper, but it was not until recently that its crucial role has been put forward [25,26]. As it was shown therein, the resonant helicity mixing condition is also defined in terms of material constants, i.e., it is fulfilled in samples with matched refractive indices. Moreover, it permits the construction of spherical scatterers which flip helicity of light very efficiently, while still respecting the energy conservation law. Finally, also in line with impedance-matched materials, index-matched materials have an associated conserved quantity: the square of linear momentum. As we show next, the conservation of this magnitude can be related with a Casimir invariant and, thus, its comprehension requires a systematic approach to the description of electromagnetic waves in terms of group theory.

In this work, we first revisit the theory of unitary irreducible representations of the Poincaré group and its link to the description of electromagnetic waves in vacuum. We place at the forefront of our analysis, the important group theoretical concept of Casimir operators, applied to some problems in electromagnetism. In group theory, the Casimir operators of a particular continuous group are operators that commute with all the generators and, thus, they label the unitary irreducible representations. In this line, we show that, exactly as Bialynicki-Birula's photon wave function [27] is associated with the unitary irreducible representations of the Poincaré group [28], monochromatic electromagnetic waves propagating in infinitely homogeneous media are associated with the unitary irreducible representations of the $P_{3,1}$ group. This is a subgroup of the Poincaré group which can be decomposed as the direct product of the Euclidean group in three dimensions, $E(3)$, and the one-parameter group of time translations, T. As a result, electromagnetic wave solutions propagating in infinitely homogeneous media can be constructed as eigenfunctions of the three Casimir operators of $P_{3,1}$, i.e., the generator of time translations, $\hat{P}_0$, the helicity operator, $\hat{\Lambda}$, and the square of the linear momentum operator, $\hat{\mathbf{P}}^2$. Finally, we study the propagation of electromagnetic waves in inhomogeneous media and we particularly focus on Kerker's problem, i.e., the scattering of electromagnetic waves by magnetic spheres. We show that the two matching conditions reported in the seminal paper are related to the breaking of the homogeneity of space, while still preserving the Casimir invariants of $P_{3,1}$.

II. ELECTROMAGNETIC WAVES IN VACUUM

One of the most elemental descriptions of the propagation of electromagnetic waves in vacuum is due to Eugene Wigner [29,30]. His findings not only provided an axiomatic derivation of Maxwell's equations in vacuum, but unified the description of all relativistic particles in terms of the so-called Wigner's classification. His major achievement is probably the connection between unitary irreducible representations (UIRs) of the Poincaré group and wave functions of isolated physical systems in vacuum. Indeed, he found that the invariant vector spaces of the different UIRs of the Poincaré group are associated with the wave functions of relativistic particles [31]. More specifically, Wigner and Bargmann found that the UIRs of the Poincaré group (and, in fact, of many other continuous groups) can be labeled by the eigenvalues of the two principal Casimir operators: $\hat{C}_1 = \hat{P}_\nu \hat{P}^\nu$ and $\hat{C}_2 = \hat{W}_\nu \hat{W}^\nu$, which are the modulus square of the four momentum, $\hat{P}_\nu$, and the modulus square of the Pauli-Lubanski pseudovector, $\hat{W}_\nu$, respectively [30,32]. In practice, the first Casimir operator represents the mass, whereas the second Casimir operator is associated with the internal degrees of freedom of the particles. In this section, we will mostly focus on electromagnetic waves propagating in vacuum and, thus, we will center our analysis on the massless and discrete-spin UIRs of the Poincaré group. For this type of particle, a third Casimir operator is identified [33]: helicity, $\hat{\Lambda}$.

The description of the photon, as a fundamental relativistic particle, is considered within the 0_s class of UIRs of the Poincaré group [30]. This class describes particles of null mass, $M = 0$, and helicity eigenvalue $\lambda = \pm S$, where S is an integer or an odd-half-integer [31]. The electromagnetic case is recovered when fixing $S = 1$. The invariant vector spaces associated with the description of the photon fulfill the following relations: $\hat{C}_1 = 0$ and $\hat{W}_\nu = \lambda \hat{P}_\nu$ [30]. Even if the form might be cumbersome, it can be shown that the first equation represents the electromagnetic wave equation in vacuum. Moreover, the time component of the second equation represents Faraday-Ampère's laws in vacuum and, finally, Gauss' laws can be derived from the spatial components of the second equation [34–36].

In our view, the deepest explicit connection between the UIRs of the Poincaré group and Maxwell's equations has been carried out by Bialynicki-Birula and Bialynicka-Birula. In their works, they argue that the electromagnetic object which more closely follows the UIRs of the Poincaré group is the Riemann-Silberstein (RS) vector, from which a proper photon wave function can be constructed. Explicitly, they have proposed the following form of the photon wave function in vacuum [28]:

$$\mathbf{\Psi}^\lambda(\mathbf{r}, t) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k}\, f^\lambda(\mathbf{k}) \mathbf{e}^\lambda(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_\mathbf{k} t)}, \quad (1)$$

where $\mathbf{r}$ is the position vector and t the time. On the other hand, $\lambda = \pm 1$ is the helicity label, $\mathbf{k}$ is the wave vector of the radiation field, $f^\lambda(\mathbf{k})$ is an arbitrary complex amplitude associated with a fixed wave vector and helicity, $\mathbf{e}^\lambda(\mathbf{k})$ is a unitary polarization vector, and $\omega_\mathbf{k} = |\mathbf{k}|$ is the angular frequency (we choose natural units, $\hbar = c = 1$). Finally, the integral in Eq. (1) is considered over the entire reciprocal space.

The expression in Eq. (1) of the photon wave function has been derived as a particular solution of Maxwell's equations in terms of the RS vector. However, let us now show that Eq. (1) can also be derived by constructing the invariant vector spaces of the massless UIRs of the Poincaré group [37,38]. This connection highlights that important features of electromagnetic wave solutions in vacuum can be derived from pure group theoretical arguments. In this regard, we closely follow the method indicated by Tung's book, *Group Theory in Physics* [31], which indicates that the construction of invariant spaces can be carried out by operating over a "standard" vector. Indeed, the basis vectors of the massless and discrete-spin

UIRs of the Poincaré group are constructed as (see Ref. [31], Chap. 10, Sec. 4):

$$\boldsymbol{\Psi}^{\lambda}_{\mathbf{k}}(\mathbf{r},t) \equiv [\hat{R}_z(\phi)\hat{R}_y(\theta)\hat{L}_z(\xi)]\boldsymbol{\Psi}^{\lambda}_{\mathbf{k}_l}(\mathbf{r},t), \tag{2}$$

where $\boldsymbol{\Psi}^{\lambda}_{\mathbf{k}_l}(\mathbf{r},t) = |\mathbf{k}_l|\mathbf{u}^{\lambda}e^{i|\mathbf{k}_l|(z-t)}$ is a monochromatic plane wave propagating in the positive OZ direction with wave vector $\mathbf{k}_l$, and $\mathbf{u}^{\lambda} = (1, \lambda i, 0)/\sqrt{2}$ is a polarization vector with well-defined helicity λ. $\hat{R}_z(\phi)$ represents a rotation along the OZ axis by an angle ϕ and $\hat{R}_y(\theta)$ represents a rotation along the OY axis by an angle θ. On the other hand, $\hat{L}_z(\xi)$ represents a Lorentz transformation along the OZ direction to an inertial frame moving with velocity $v = \tanh(\xi)$. In addition, note that the introduction of the amplitude factor $|\mathbf{k}_l|$ in the definition of the standard vector, $\boldsymbol{\Psi}^{\lambda}_{\mathbf{k}_l}(\mathbf{r},t)$, makes it transform unitarily under Lorentz transformations [39].

The application of the $\hat{L}_z(\xi)$ operator modifies the wavenumber of the standard vector as follows: $|\mathbf{k}| = e^{-\xi}|\mathbf{k}_l|$. Thus, by choosing the boost parameter $\xi \in (-\infty, \infty)$, one spans all possible values of the wave vector modulus, i.e., $|\mathbf{k}| \in (0, \infty)$ (see Appendix A). On the other hand, by choosing $\phi \in (0, 2\pi)$ and $\theta \in (0, \pi)$ the rotation transformation makes the monochromatic plane wave propagate in an arbitrary direction. If we now consider the integral in Eq. (1) in spherical coordinates of reciprocal space, we see that the three parameters $\{\xi, \theta, \phi\}$ span all the integration domain. In other words, the photon wave function in vacuum specified in Eq. (1) can be written in the following form:

$$\boldsymbol{\Psi}^{\lambda}(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}}\int \frac{d\mathbf{k}}{|\mathbf{k}|} f^{\lambda}(\mathbf{k})\boldsymbol{\Psi}^{\lambda}_{\mathbf{k}}(\mathbf{r},t). \tag{3}$$

Note that, even if the standard vector is a monochromatic wave, the state above is no longer monochromatic. From the result in Eq. (3) we conclude that one can retrieve the expression of the photon wave function in vacuum, in the form proposed by Bialynicki-Birula, just by taking linear superpositions of the basis vectors of the massless and discrete-spin UIRs of the Poincaré group. Now, the question arises: following similar group theoretical arguments, can we study the propagation of electromagnetic waves in a different medium other than vacuum?

III. ELECTROMAGNETIC WAVES IN INFINITELY HOMOGENEOUS MEDIA

Let us first consider the next simplest case, i.e., a medium where both the electric permittivity, ε, and magnetic permeability, μ, are different from the values of vacuum but are constant functions of space coordinates. If such a medium extends all over the space, we say that it is infinitely homogeneous. Moreover, we will also consider that the permittivity and permeability are scalar functions (isotropic) and that they do not change on time (static).

The key modification of the problem when studying the propagation of electromagnetic waves in an infinitely homogeneous medium compared to the case of vacuum is the set of underlying symmetries. An infinitely homogeneous medium is not invariant under the Poincaré group. This is due to the fact that an isotropic medium becomes, in general, bianisotropic when performing a Lorentz transformation

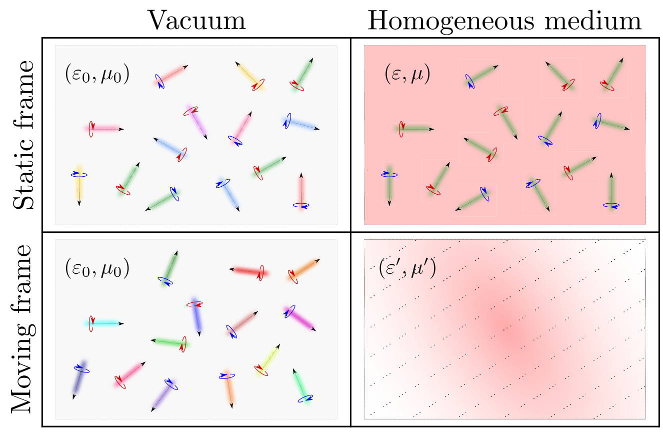

FIG. 1. Effect of a Lorentz transformation over electromagnetic waves propagating in vacuum and in a homogeneous medium. Black arrows represent the propagation directions, (θ, ϕ), red/blue spinning arrows show the helicity, λ, and different colors indicate the frequency of the waves, ω. In the left upper panel, a superposition of plane waves of different frequencies propagating in vacuum is shown. When switching to the moving reference frame (left bottom panel), the environment is left unaltered and the waves change their frequency. In the right upper panel, a superposition of monochromatic plane waves in a homogeneous medium is shown. When switching to the moving reference frame (right bottom panel), the waves experience an effectively modified medium with bianisotropic optical response.

(see Fig. 1) [40,41]. As a result, the presence of a material reduces the symmetry group from the Poincaré group to a subgroup that does not contain Lorentz transformations. Such a particular way of addressing physical problems is commonly denoted as the symmetry-breaking principle and it can be compactly stated in the following terms [42–48]: consider a physical system which is described by a given group G, and an external influence reduces the symmetry from the original G to a subgroup $G_i \subset G$. Then, the subgroup G_i can be used to study the properties of the modified system. In particular, the generators and Casimir operators of the subgroup will provide conserved quantities and the UIRs will determine the new wave functions or, at least, some of their properties.

The symmetry-breaking principle was originally proposed to simplify the solution of quantum mechanical systems in particular environments with continuous symmetries [42]. In addition to this, the principle has also been employed in the context of discrete symmetries and space groups. This approach has been implemented both in the context of condensed matter physics and photonic crystals [49]. However, the symmetry-breaking principle and Casimir invariants have not frequently been jointly discussed in the framework of electromagnetic waves and material environments with continuous space-time symmetries. In what follows, we show that the application of these concepts to electromagnetic waves propagating in continuous media also leads to fundamental results.

Let us apply the symmetry-breaking principle to study the propagation of electromagnetic waves in an infinitely homogeneous medium. As stated above such a medium is not invariant under the full Poincaré group, but only under the subgroup $P_{3,1}$ which includes all the generators of the Poincaré group except for Lorentz boosts [48]. In other words, $P_{3,1}$ comprises seven generators: the generator of time translations ($\hat{P}_0$), the three generators of space translations (linear momentum components, $\hat{P}_i$), and the three generators of spatial rotations (total angular momentum components, $\hat{J}_i$). In addition to this, the subgroup has three Casimir operators, i.e., the generator of time translations, $\hat{P}_0$, a magnitude proportional to the helicity, $\hat{\mathbf{J}} \cdot \hat{\mathbf{P}}$, and the square of linear momentum, $\hat{\mathbf{P}}^2$. Helicity and square of linear momentum are conserved magnitudes for electromagnetic waves propagating in infinitely homogeneous media due to their condition of Casimir invariants. These invariants will be of utmost importance in Secs. IV and V, when discussing the conserved quantities associated with the Kerker phenomena.

Casimir operators play a central role in the determination of the UIRs. This is because the basis vectors of the UIRs are necessarily eigenvectors of all such operators [30–32]. Thus, for the basis vectors associated with the UIRs of the $P_{3,1}$ subgroup, $\mathbf{\Phi}^\lambda_{\mathbf{k}}(\mathbf{r}, t)$, we have that

$$\hat{P}_0\, \mathbf{\Phi}^\lambda_{\mathbf{k}}(\mathbf{r}, t) = \omega \mathbf{\Phi}^\lambda_{\mathbf{k}}(\mathbf{r}, t), \tag{4}$$

$$k^{-1}\hat{\mathbf{J}} \cdot \hat{\mathbf{P}}\, \mathbf{\Phi}^\lambda_{\mathbf{k}}(\mathbf{r}, t) = \lambda \mathbf{\Phi}^\lambda_{\mathbf{k}}(\mathbf{r}, t), \tag{5}$$

$$\hat{\mathbf{P}}^2\, \mathbf{\Phi}^\lambda_{\mathbf{k}}(\mathbf{r}, t) = k^2 \mathbf{\Phi}^\lambda_{\mathbf{k}}(\mathbf{r}, t). \tag{6}$$

Setting $\lambda = \pm 1$ and $k = \omega n$, with $n = \sqrt{\varepsilon \mu}$ the refractive index of the medium, it can be noted that Eqs. (4)–(6) actually represent the dynamic equations of monochromatic electromagnetic waves propagating in an infinitely homogeneous medium (see Appendix B). Indeed, Eq. (4) determines the monochromaticity of the fields; on the other hand, Eq. (5) represents all four time-independent Maxwell's equations; and, finally, Eq. (6) represents the wave equation for monochromatic waves, i.e., Helmholtz's equation. The connection appears more clearly when employing the explicit form of the operators [50–53]: $\hat{P}_0 = i\partial_t$, $\hat{\Lambda} = k^{-1}\hat{\mathbf{J}} \cdot \hat{\mathbf{P}} = k^{-1}\nabla \times$ and $\hat{\mathbf{P}}^2 = -\nabla^2$. This is in agreement with the results previously obtained for vacuum, where Casimir operators were also related with the dynamic equations of electromagnetic waves.

At this stage, following the same procedure as in the case of vacuum, we should be able to obtain some information of the photon wave function in an infinitely homogeneous medium by analyzing the UIRs of $P_{3,1}$. Particularly, $P_{3,1}$ can be split as the direct product of two of its subgroups which are the Euclidean group in three dimensions, $E(3)$, and the one-parameter subgroup of time translations, T. This is due to the fact that time translations commute with all the elements of the Euclidean group in three dimensions. As a result, the UIRs of $P_{3,1}$ are constructed as the product of the UIRs of $E(3)$ and the UIRs of T [54,55]. An intuitive way of understanding this is by noting that the basis vectors $\mathbf{\Phi}^\lambda_{\mathbf{k}}(\mathbf{r}, t)$ are necessarily built as the product of a spatial function and a temporal function. Equations (5) and (6) determine the spatial dependence of the vectors, whereas Eq. (4) determines their temporal dependence. Thus, the basis vectors of the UIRs of $P_{3,1}$ associated with electromagnetic waves can be constructed in the following way (see Ref. [31], Chap. 9, Sec. 7):

$$\mathbf{\Phi}^\lambda_{\mathbf{k}}(\mathbf{r}, t) \equiv [\hat{R}_z(\phi)\hat{R}_y(\theta)]\mathbf{\Phi}^\lambda_{\mathbf{k}_0}(\mathbf{r})e^{-i\omega t}, \tag{7}$$

TABLE I. Behavior of the photon wave functions in vacuum, $\mathbf{\Psi}^\lambda(\mathbf{r}, t)$, and in a homogeneous medium, $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$, under different Casimir operators. The check mark indicates that the wave function is an eigenvector of the indicated Casimir operator. $\hat{C}_0 = \hat{P}_0/|\hat{P}_0|$ is the sign of frequency, $\hat{C}_1 = \hat{\mathbf{P}}^2 - \hat{P}_0^2$ is the modulus square of the four momentum, and $\hat{C}_2 = \hat{\mathbf{W}}^2 - (\hat{\mathbf{J}} \cdot \hat{\mathbf{P}})^2$ the modulus square of the Pauli-Lubanski pseudovector. $\hat{\mathbf{W}} = \hat{P}_0\hat{\mathbf{J}} + \hat{\mathbf{K}} \times \hat{\mathbf{P}}$ is the spatial component of the Pauli-Lubanski pseudovector, with $\hat{\mathbf{K}}$ the boost operator, i.e., the generator of Lorentz transformations. Finally, note that $\hat{P}_0$ and $\hat{\mathbf{P}}^2$ operators label different UIRs of the $P_{3,1}$ subgroup, whereas Poincaré UIRs are superpositions of monochromatic waves with different frequencies.

	$\hat{C}_0$	$\hat{C}_1$	$\hat{C}_2$	$\hat{\Lambda}$	$\hat{P}_0$	$\hat{\mathbf{P}}^2$
$\mathbf{\Psi}^\lambda(\mathbf{r}, t)$	✓	✓	✓	✓		
$\mathbf{\Phi}^\lambda(\mathbf{r}, t)$	✓	✓		✓	✓	✓

where $\mathbf{\Phi}^\lambda_{\mathbf{k}_0}(\mathbf{r}) = \mathbf{u}^\lambda e^{ikz}$ is now the standard vector and $\mathbf{u}^\lambda = (1, \lambda i, 0)/\sqrt{2}$ is a circular polarization vector.

Note that the basis given in Eq. (7) bears a close resemblance to the basis previously constructed for the UIRs of the Poincaré group. Indeed, they are constructed exactly in the same way except for the Lorentz boost applied in Eq. (2), which permits the modulation of the frequency through parameter ξ. The basis vectors given by Eq. (7), on the other hand, have a fixed frequency ω (see Table I). As a result, following the symmetry-breaking principle, we conclude that the photon wave function in an infinitely homogeneous medium should be obtained as

$$\mathbf{\Phi}^\lambda(\mathbf{r}, t) = \int d\Omega\, \varphi^\lambda(\Omega)\mathbf{\Phi}^\lambda_{\mathbf{k}}(\mathbf{r}, t), \tag{8}$$

where $d\Omega = \sin\theta d\theta d\phi$, with integration limits $\phi \in (0, 2\pi)$ and $\theta \in (0, \pi)$, and $\varphi^\lambda(\Omega)$ is an arbitrary complex function of the azimuthal and polar angles. Note that such a field is nothing but an abstract representation of the monochromatic RS vector (see Appendix B). As a matter of fact, such a way of expressing electromagnetic fields has already been employed to construct wave solutions with well-defined helicity, in particular, vector spherical harmonics and Bessel beams [31,51]. The way in which we have derived it here indicates that the validity of the expression lies in the underlying symmetry group and that it goes far beyond the specific examples previously reported.

Our findings indicate that the monochromatic RS vector, $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$, is intimately linked with the $P_{3,1}$ subgroup and, thus, it plays a central role in the description of optical phenomena in material environments. Furthermore, we have shown that the dynamic equations that are employed to study the propagation of electromagnetic waves propagating in infinitely homogeneous media can be derived from pure group theoretical arguments. In particular, we have highlighted the fundamental role that the Casimir operators $\hat{P}_0$, $\hat{\Lambda}$, and $\hat{\mathbf{P}}^2$ play in the description of monochromatic electromagnetic waves. Also, note that the symmetry-breaking principle had previously been applied for nonrelativistic massive particles [43,47]. Indeed, the time-independent Schrödinger's equation in an infinitely homogeneous medium can also be

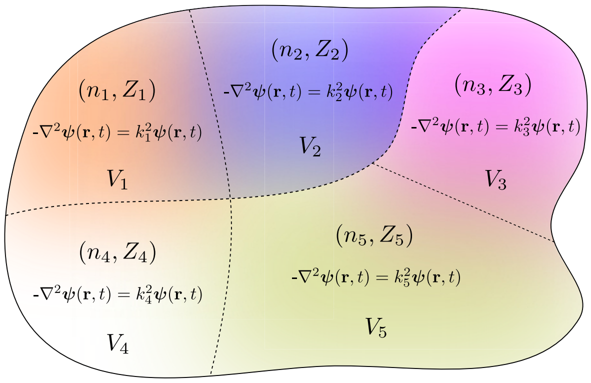

FIG. 2. Sketch of a piecewise homogeneous medium. Electromagnetic wave solutions are obtained by solving Helmholtz's equation and applying boundary conditions at the interfaces.

associated with Eqs. (4)–(6) simply by fixing $\lambda = 0$ and $k = \sqrt{2m(E - V)}$, where E is the eigenvalue of the $\hat{P}_0$ operator and V represents a constant potential [43,47,54,56]. This mathematical analogy between the dynamic equations and wave functions describing massless and massive particles will aid us later in the discussion of the Kerker phenomena.

IV. ELECTROMAGNETIC WAVES IN PIECEWISE MEDIA: IMPEDANCE AND REFRACTIVE INDEX MATCHING

In the previous section, we have shown that the monochromatic RS vector is intimately associated with the UIRs of the $P_{3,1}$ subgroup. Indeed, following the symmetry-breaking principle, we have found that the photon wave function in an infinitely homogeneous medium should be associated with the $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$ field in Eq. (8). In practical terms, this indicates that the monochromatic RS vector is a natural choice to express electromagnetic fields in material environments. In the following, we show that the relevance of the $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$ field goes far beyond infinitely homogeneous media. We focus on piecewise homogeneous systems as it is the type of environment in which Kerker phenomena were first identified [1,2].

Piecewise homogeneous media are inhomogeneous optical environments that are constituted of different homogeneous patches (see Fig. 2). Here, we assume that those patches are built from materials whose responses do not change in time. In this case, there is only one symmetry left in the system, i.e., the one-parameter subgroup of time translations, T. It is so because piecewise environments are not generally invariant under spatial translations and rotations in three dimensions. As a result, the only symmetry of a generic piecewise medium is due to the static nature of the patches. Applying again the symmetry-breaking principle, we can state a few things about the dynamics in static piecewise media, i.e., that frequency, ω, is conserved and that the electromagnetic wave solutions are eigenstates of $\hat{P}_0$. The UIRs of the group of time translations can be labeled as (see Ref. [31], Chap. 6, Sec. 6)

$$\hat{P}_0 \boldsymbol{\psi}(\mathbf{r}, t) = \omega \boldsymbol{\psi}(\mathbf{r}, t), \tag{9}$$

where $\boldsymbol{\psi}(\mathbf{r}, t)$ represents an electromagnetic wave solution in a generic piecewise medium. This condition is equivalent to Eq. (4). And, as before, it implies that the time dependence of the solutions in piecewise homogeneous media is fixed and it is given by a complex exponential.

Strictly speaking, the information provided by the application of the symmetry-breaking principle to generic piecewise media is fully contained in Eq. (9). In other words, if we exclusively rely on rigorous group theoretical arguments, we can just state that solutions in static piecewise media are eigenstates of $\hat{P}_0$ operator. However, the analysis of previous sections provides us with a deeper insight that we can employ to study the propagation of electromagnetic waves in piecewise media. In particular, we have seen that the monochromatic RS vector, $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$, naturally emerges within the UIRs of the $P_{3,1}$ subgroup and, thus, in the context of electromagnetic waves propagating in material environments (see Appendix B). Taking into account the fundamental grounds in which this field arises, we may consider expressing Maxwell's equations in inhomogeneous media also in terms of the $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$ field (see Appendix C). As we show next, by doing so, the matching conditions reported in Secs. II and IV of Kerker's original paper emerge in a completely natural manner.

In a generic inhomogeneous medium, Maxwell's equations can be expressed as (see Appendix C) [25,57]

$$i\partial_t \mathbf{\Phi}^+ = \frac{1}{\sqrt{n}} \nabla \times \left(\frac{\mathbf{\Phi}^+}{\sqrt{n}} \right) + \frac{1}{n} \nabla \ln \sqrt{Z} \times \mathbf{\Phi}^-, \tag{10}$$

$$i\partial_t \mathbf{\Phi}^- = -\frac{1}{\sqrt{n}} \nabla \times \left(\frac{\mathbf{\Phi}^-}{\sqrt{n}} \right) - \frac{1}{n} \nabla \ln \sqrt{Z} \times \mathbf{\Phi}^+, \tag{11}$$

where, for clarity, we have omitted the dependence on the position vector, $\mathbf{r}$, and time, t. Also, we have defined the local impedance $Z(\mathbf{r}) = \sqrt{\mu(\mathbf{r})/\varepsilon(\mathbf{r})}$ and local refractive index $n(\mathbf{r}) = \sqrt{\varepsilon(\mathbf{r})\mu(\mathbf{r})}$. Given the specific form that Maxwell's equations adopt in terms of the monochromatic RS vector, $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$, we can identify two types of environments in which the electromagnetic field has a particular behavior: media for which the impedance is constant ($\nabla Z = 0$) and media for which the refractive index is constant ($\nabla n = 0$). We usually denote these two situations as the impedance matching and the refractive index matching conditions, respectively, and they are discussed in Secs. II and IV of Kerker's paper [1]. Note that, in piecewise environments, the impedance matching condition is fulfilled when $Z_j = \sqrt{\mu_j/\varepsilon_j}$ is constant in all domains V_j. On the other hand, the refractive index matching condition is fulfilled whenever $n_j = \sqrt{\varepsilon_j \mu_j}$ is constant in all domains V_j.

In this line, Fernandez-Corbaton and coworkers indicated that the impedance matching condition leads to the conservation of electromagnetic helicity [21]. It was shown that, whenever μ_j/ε_j is constant, Maxwell's equations in the whole piecewise medium remain invariant under electromagnetic duality transformations. As helicity had previously been identified as the generator of the duality transformation [20], it was then concluded that helicity had to be preserved in an impedance-matched piecewise homogeneous medium. However, there is also a dynamical way of understanding the conservation of helicity in impedance-matched media. Indeed, given the form of Maxwell's equations specified in Eqs. (10) and (11), it can be checked that the impedance matching condition makes the two helicity components of the electromagnetic field to be decoupled. This implies that, if we fix the initial conditions at time $t = 0$ to contain a single helicity component, the solutions in an impedance-matched piecewise medium will also contain a single helicity component at times $t > 0$. As a result, in the case of piecewise media, the solutions under the impedance matching condition, $\boldsymbol{\psi}_\lambda(\mathbf{r}, t)$, fulfill $\hat{\Lambda}\boldsymbol{\psi}_\lambda(\mathbf{r}, t) = \lambda\boldsymbol{\psi}_\lambda(\mathbf{r}, t)$, with $\lambda = \pm 1$. Thus, we can also infer the conservation of helicity in piecewise media from the fact that eigenstates of $\hat{\Lambda}$, acting as the Casimir operator in Eq. (5), remain eigenstates of $\hat{\Lambda}$ in dual media.

In this sense, the helicity operator has a double role. On the one hand, it is the generator of dual transformations, but as seen in Eq. (5), it also acts as a Casimir operator. This could raise the question of whether Casimir operators are of some interest to study electromagnetic wave dynamics in piecewise homogeneous media. We will show now that, in the case of index-matched piecewise media, the conserved quantity is not associated with the generator of a continuous symmetry transformation, but with another Casimir operator $\hat{\mathbf{P}}^2$.

Under the refractive index matching condition, the conservation of a physical magnitude can be inferred from how electromagnetic wave solutions are built in piecewise media. Indeed, as it is shown in Fig. 2, solutions of electromagnetic waves propagating in this type of environment are constructed by solving Helmholtz's equation in each domain V_j and, then, applying boundary conditions. This implies that, in a generic piecewise homogeneous medium, electromagnetic wave solutions fulfill $-\nabla^2\boldsymbol{\psi}(\mathbf{r}, t) = k_j^2\boldsymbol{\psi}(\mathbf{r}, t)$ for $\mathbf{r} \in V_j$, where $k_j = \omega n_j$. Note that, for a generic piecewise homogeneous medium, k_j changes depending on the region of space we may consider. Thus, the eigenvalue of the $\hat{\mathbf{P}}^2 = -\nabla^2$ operator varies from one region V_j to another $V_{j'}$, as long as $n_j \neq n_{j'}$. However, whenever the refractive index matching condition is fulfilled ($n_j = n_{j'}, \forall j, j'$), the wave vector modulus is constant all over the medium and the square of linear momentum operator fulfills $\hat{\mathbf{P}}^2\boldsymbol{\psi}_k(\mathbf{r}, t) = k^2\boldsymbol{\psi}_k(\mathbf{r}, t)$, where now k is fixed and $\boldsymbol{\psi}_k(\mathbf{r}, t)$ represents wave solutions for index-matched piecewise media. Note that this way of representing the conservation of the square of linear momentum is exactly the same as in Eq. (6). The refractive index matching condition recovers that same relation but in piecewise environments.

Following the previous discussion on helicity conservation, we may try to link the conservation of $\hat{\mathbf{P}}^2$ with the restoration of a symmetry. This, however, is not possible because the square of linear momentum is not a generator of any continuous symmetry transformation. Therefore, we are led to comprehend the conservation of $\hat{\mathbf{P}}^2$ from its condition of Casimir invariant. Indeed, Casimir operators of a group G may remain as conserved quantities in environments whose symmetry group is a subgroup $\mathrm{G}_i \subset G$ [48]. This phenomenon is not a particularity of electromagnetic waves, it is known to occur also in the dynamics of massive particles. Let us put forward a particular example that very closely mimics the refractive index matching condition for electromagnetic waves. As we have shown in Sec. III, the conservation of $\hat{\mathbf{P}}^2$ for nonrelativistic massive particles can also be related to the symmetries of Euclidean space [54]. Thus, in principle, we

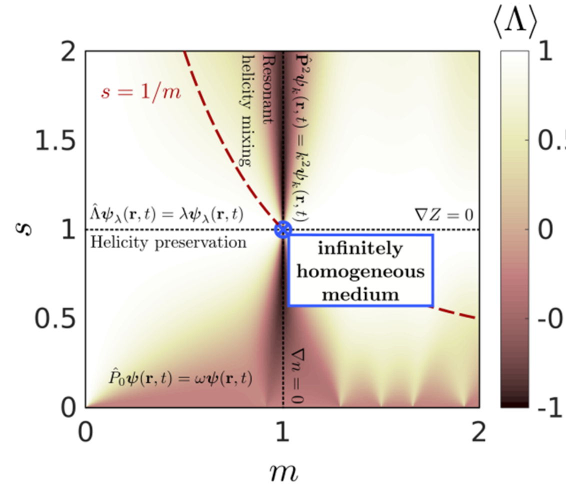

FIG. 3. Emergence of the Kerker phenomena in a magnetic sphere embedded in a homogeneous surrounding. The helicity expectation value, $\langle\Lambda\rangle$, quantifies the helicity of the scattered electromagnetic field. s is the impedance contrast and m the refractive index contrast. Frequency, ω, is preserved at every point due to the static nature of the piecewise system. Helicity is preserved whenever the impedance is constant (horizontal dashed line). Square of linear momentum is preserved whenever the refractive index is constant (vertical dashed line). The singular point $[m, s] = [1, 1]$ represents an infinitely homogeneous medium.

may expect the square of linear momentum to be exclusively preserved when massive particles propagate in free space. This, however, is not true. There is a particular interaction potential, i.e., the hard-sphere potential, for which $\hat{\mathbf{P}}^2$ is a conserved quantity (see Appendix D). The same also holds for classical mechanics, where the kinetic energy, proportional to the square of the linear momentum, is known to be preserved provided that the interactions are elastic.

V. SCATTERING OF ELECTROMAGNETIC WAVES BY MAGNETIC SPHERES

Finally, let us apply the previous analysis on conserved quantities to a particular case of special significance, i.e., the emergence of the Kerker phenomena in magnetic spheres. To be clear in the wording, instead of the usual phrase Kerker conditions, we employ Kerker phenomena to denote both the impedance and refractive index matching conditions in magnetic spheres.

In Fig. 3, we show the analysis of conserved quantities associated with the helicity map of a magnetic sphere with size parameter $x = 3$ (see Appendix E for details on numerical calculations). Let us briefly remind that $\langle\Lambda\rangle$ represents the helicity expectation value, which is an observable that determines whether electromagnetic helicity of the incident wave is conserved, $\langle\Lambda\rangle = 1$, or completely flipped, $\langle\Lambda\rangle = -1$, upon scattering [24]. Along the vertical axis we tune the impedance contrast, s, which is the ratio between the impedances of the sphere and the surrounding medium. In the horizontal axis, we fix the refractive index contrast, m, which is the ratio between the refractive indices. Due to the static nature of the media involved, all the solutions represented in the figure are eigenstates of $\hat{P}_0$. Then, under the impedance matching condition, along the line $s = 1$, helicity is conserved and, thus, solutions are eigenstates of $\hat{\Lambda}$. In addition to this, under the index matching condition, whenever $m = 1$, square of linear momentum is conserved and, as a result, solutions are eigenstates of the $\hat{\mathbf{P}}^2$ operator. Note that, whenever $m = 1$, the helicity of the scattered field is almost completely flipped (see Fig. 3) and, thus, this condition has been denoted as the resonant helicity mixing condition [25,26]. On the other hand, the line $s = 1/m$ represents the response of nonmagnetic materials, i.e., those for which $\mu = 1$. Finally, in the singular point $[m, s] = [1, 1]$ of the colormap, solutions are eigenstates of all three operators $\hat{P}_0$, $\hat{\Lambda}$, and $\hat{\mathbf{P}}^2$. This is because the particular case $m = s = 1$ represents an infinitely homogeneous medium and, by means of Eqs. (4)–(6), electromagnetic waves are constructed as eigenstates of the three Casimir operators of $P_{3,1}$.

The Kerker phenomena can also be interpreted by making use of the form of Maxwell's equations when they are expressed in terms of the monochromatic RS vector, $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$. Indeed, under both matching conditions, i.e., $\nabla Z = 0$ and $\nabla n = 0$, the environment ceases to be inhomogeneous and it becomes infinitely homogeneous. In this situation, Maxwell's equations as expressed in Eqs. (10) and (11) converge to the analytical form dictated by Eqs. (4)–(6). In other words, electromagnetic wave solutions are constructed as eigenstates of the Casimir operators $\hat{P}_0$, $\hat{\Lambda}$, and $\hat{\mathbf{P}}^2$. This situation is equivalent to the one represented by the $[m, s] = [1, 1]$ point in Fig. 3. Then it is clear that Eqs. (10) and (11) indicate two preferential directions in which the homogeneity of space may be broken. These two preferential directions are associated with the two matching conditions. On the one hand, the homogeneity of space may be broken while still keeping the impedance constant ($\nabla Z = 0$). In piecewise homogeneous media, this leads to the conservation of both $\hat{P}_0$ and $\hat{\Lambda}$ and the situation is equivalent to the $s = 1$ line indicated in Fig. 3. On the other hand, the homogeneity of space may also be broken while keeping the refractive index constant ($\nabla n = 0$). In piecewise homogeneous media, this leads to the conservation of $\hat{P}_0$ and $\hat{\mathbf{P}}^2$ and the situation is equivalent to the $m = 1$ line indicated in Fig. 3.

In our view, this analysis provides a fundamental interpretation of Maxwell's equations in inhomogeneous media as expressed in Eqs. (10) and (11). Note that such a particular form of expressing the equations is obtained through a simple change of basis (see Appendix C). Indeed, instead of the usual electric, $\mathbf{E}(\mathbf{r}, t)$, and magnetic, $\mathbf{H}(\mathbf{r}, t)$, fields we rewrite Maxwell's equations in terms of the monochromatic RS vector, $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$, which we have shown to be linked with the UIRs of the $P_{3,1}$ subgroup (see Sec. III). As a result, we obtain a completely equivalent form of the equations in which instead of the usual material parameters $\varepsilon(\mathbf{r})$ and $\mu(\mathbf{r})$, derivatives of the local impedance, $Z(\mathbf{r})$, and refractive index, $n(\mathbf{r})$, appear. Finally, we find that these material parameters are closely related to the Casimir invariants of $P_{3,1}$. It really seems like the connection of Maxwell's equations with space-time symmetries is clarified when expressing them in terms of the monochromatic RS vector, $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$. In this line, the Kerker phenomena represent an expression of such a fundamental connection in a particular electromagnetic scattering problem.

VI. CONCLUSION

In conclusion, we have revisited the link of Maxwell's equations with the theory of representations of continuous groups. Starting from the Poincaré group, we have shown that its subgroups may be employed to analyze conserved quantities and wave solutions in different material environments. This procedure, which had previously been employed in the framework of quantum mechanics, is systematically applied to the study of classical electromagnetic waves in continuous media.

In this context, we have obtained a set of fundamental results. First, we have revisited the notion of the photon wave function in vacuum, $\mathbf{\Psi}^\lambda(\mathbf{r}, t)$, and derived it based on pure group theoretical arguments [37,38]. Second, we have shown that time-independent Maxwell's equations in infinitely homogeneous media can be derived from the Casimir invariants of the $P_{3,1}$ subgroup of the Poincaré group. As a result, we have identified the fundamental role that the monochromatic RS vector, $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$, plays in the description of electromagnetic waves propagating in matter. Third, we have seen that the impedance and refractive index matching conditions in piecewise media are associated with the conservation of two Casimir invariants of $P_{3,1}$, i.e., helicity and square of linear momentum. Finally, we have shown that Kerker phenomena constitute an example of how the matching conditions emerge in a particular scattering problem.

Our contribution not only gives a sound mathematical foundation to the origin of the Kerker phenomena but also invites the search of new effects based on similar group theoretical arguments. In addition, we have identified the monochromatic RS vector, $\mathbf{\Phi}^\lambda(\mathbf{r}, t)$, as the fundamental object associated with the description of electromagnetic waves propagating in matter. This may be useful to obtain more insightful descriptions of optical phenomena in piecewise homogeneous systems.

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APPENDIX A: BOOSTS ACTING OVER A MONOCHROMATIC PLANE WAVE WITH WELL-DEFINED HELICITY

In this Appendix, we analyze the action of the Lorentz boost in the OZ direction, $\hat{L}_z(\xi)$, over the standard vector as expressed in Eq. (2). In general, if a boost is carried out in the $\mathbf{b}/|\mathbf{b}|$ direction, a generic electromagnetic field with well-defined helicity, $\mathbf{V}_\pm$, is transformed as [58,59]

$$\mathbf{V}'_\pm = \gamma(\mathbf{V}_\pm \mp i\mathbf{b}\times\mathbf{V}_\pm) - \frac{\gamma^2}{\gamma+1}\mathbf{b}(\mathbf{b}\cdot\mathbf{V}_\pm), \quad \text{(A1)}$$

where $|\mathbf{b}| = v$ and $\gamma = (1-|\mathbf{b}|^2)^{-1/2}$. Here, we are interested in the transformation properties of the field $\mathbf{V}_\pm = \mathbf{u}_\pm e^{i|\mathbf{k}_l|(z-t)}$, with $\mathbf{u}_\pm = (1, \pm i, 0)$. Moreover, as the boost is along the OZ axis, we need to fix $\mathbf{b}/|\mathbf{b}| = \hat{z}$ and, as a result, we have that $i\mathbf{b}\times\mathbf{u}_\pm = \pm|\mathbf{b}|\mathbf{u}_\pm$. It can also be checked that $\mathbf{b}$ and $\mathbf{V}_\pm$ are orthogonal, i.e., $\mathbf{b}\cdot\mathbf{V}_\pm = 0$, which makes the second term in Eq. (A1) vanish. Finally, applying the inverse Lorentz transformation to the coordinates, i.e., from $(\mathbf{r}, t)$ to $(\mathbf{r}', t')$, the standard vector finally results in

$$\mathbf{V}'_\pm = \gamma(1-|\mathbf{b}|)\mathbf{u}_\pm e^{i\gamma(1-|\mathbf{b}|)|\mathbf{k}_l|(z'-t')}. \quad \text{(A2)}$$

The expression in the equation above shows that apart from changing its amplitude, the standard vector also changes the modulus of its wave vector as

$$|\mathbf{k}'| = \gamma(1-|\mathbf{b}|)|\mathbf{k}_l| = (\cosh\xi - \sinh\xi)|\mathbf{k}_l| = e^{-\xi}|\mathbf{k}_l|. \quad \text{(A3)}$$

Note that different sign conventions are considered in Tung and Jackson for the Lorentz boosts [see, for instance, Eqs. (10.1-9) in Tung and Eq. (11.21) in Jackson]. The result in Eq. (A3) implies that the boost parameter ξ controls the modulus of the wave vector, $\mathbf{k}'$, of the monochromatic plane wave for a fixed $|\mathbf{k}_l|$. As the frequency and the modulus of the wave vector are proportional, one finally gets that ω is also modulated by the parameter ξ. Importantly, note that the boost $\hat{L}_z(\xi)$ does not modify either the helicity or the direction of propagation of the standard vector because it travels along the OZ axis.

APPENDIX B: MAXWELL'S EQUATIONS IN AN INFINITELY HOMOGENEOUS MEDIUM

In this Appendix, we show that Eqs. (4)–(6) represent the monochromaticity condition, the time-independent Maxwell's equations, and Helmholtz's equations, respectively. For that aim, let us bring forward the usual definition of the monochromatic RS vector in an infinitely homogeneous medium:

$$\mathbf{F}^\lambda(\mathbf{r}, t) = \frac{1}{\sqrt{2}}\left[\frac{\mathbf{D}(\mathbf{r})}{\sqrt{\varepsilon}} + \lambda i\frac{\mathbf{B}(\mathbf{r})}{\sqrt{\mu}}\right]e^{-i\omega t}. \quad \text{(B1)}$$

Notice that the real electric displacement and magnetic induction fields can be obtained as $\mathcal{D}(\mathbf{r}, t) = \mathrm{Re}[\mathbf{D}(\mathbf{r})e^{-i\omega t}]$ and $\mathcal{B}(\mathbf{r}, t) = \mathrm{Re}[\mathbf{B}(\mathbf{r})e^{-i\omega t}]$, respectively. In this case, as the medium is infinitely homogeneous, we have that the electric permittivity, ε, and magnetic permeability, μ, are constant functions of the position vector, $\mathbf{r}$.

Given the expression in Eq. (B1) it is direct to check that such a field is an eigenstate of the $\hat{P}_0 = i\partial_t$ operator. Indeed, in line with Eq. (4) of the main text, we also have that

$$i\partial_t\mathbf{F}^\lambda(\mathbf{r}, t) = \omega\mathbf{F}^\lambda(\mathbf{r}, t). \quad \text{(B2)}$$

On the other hand, let us take the curl of the field given by Eq. (B1):

$$\begin{aligned}\nabla\times\mathbf{F}^\lambda(\mathbf{r}, t) &= \frac{1}{\sqrt{2}}[\sqrt{\varepsilon}\nabla\times\mathbf{E}(\mathbf{r}) + \lambda i\sqrt{\mu}\nabla\times\mathbf{H}(\mathbf{r})]e^{-i\omega t} \\ &= \frac{\omega}{\sqrt{2}}[i\sqrt{\varepsilon}\mathbf{B}(\mathbf{r}) + \lambda\sqrt{\mu}\mathbf{D}(\mathbf{r})]e^{-i\omega t} \\ &= \lambda\frac{\omega\sqrt{\varepsilon\mu}}{\sqrt{2}}\left[\frac{\mathbf{D}(\mathbf{r})}{\sqrt{\varepsilon}} + \lambda i\frac{\mathbf{B}(\mathbf{r})}{\sqrt{\mu}}\right]e^{-i\omega t}. \end{aligned} \quad \text{(B3)}$$

And, if we recall the definition of the wave vector modulus $k = \omega\sqrt{\varepsilon\mu}$ and the helicity operator $k^{-1}\hat{\mathbf{J}}\cdot\hat{\mathbf{P}} = k^{-1}\nabla\times$, we finally reach to the expression given in Eq. (5) of the main text. Thus, the expressions given by Eq. (B3) with $\lambda = \pm 1$ are equivalent to the time-independent Faraday-Ampère's laws in an infinitely homogeneous medium. In addition, it can also be checked that Gauss' law follows directly from them. This can be explicitly computed by taking the divergence of the expression given in Eq. (B3), which leads to $\nabla\cdot\mathbf{F}^\lambda(\mathbf{r}, t) = 0$.

Finally, we may also compute the Helmholtz's equation from the monochromatic RS vector defined in Eq. (B1). For that aim, we have to compute once again the curl over the expression given in Eq. (B3). Operating on both sides of such expression, we get

$$\nabla\times(\nabla\times\mathbf{F}^\lambda(\mathbf{r}, t)) = k^2\mathbf{F}^\lambda(\mathbf{r}, t). \quad \text{(B4)}$$

On the other hand, we may employ the vector calculus identity: $\nabla\times(\nabla\times\mathbf{V}) = \nabla(\nabla\cdot\mathbf{V}) - \nabla^2\mathbf{V}$. Note that we have previously shown that the divergence of the monochromatic RS vector vanishes. This implies that we are left with the following relation:

$$-\nabla^2\mathbf{F}^\lambda(\mathbf{r}, t) = k^2\mathbf{F}^\lambda(\mathbf{r}, t), \quad \text{(B5)}$$

which is exactly the expression previously given in Eq. (6) of the main text, as we have defined that $\hat{\mathbf{P}}^2 = -\nabla^2$. This shows that the monochromatic RS vector in an infinitely homogeneous medium also fulfills Helmholtz's equation.

We can then conclude that Eqs. (4)–(6) of the main text represent the monochromaticity condition, time-independent Maxwell's equations and Helmholtz's equation in an infinitely homogeneous medium.

APPENDIX C: MAXWELL'S EQUATIONS IN MAGNETODIELECTRIC INHOMOGENEOUS MEDIA

In this Appendix, we show the form that Maxwell's equations adopt in inhomogeneous media in terms of the monochromatic RS vector. To describe the equations in inhomogeneous media, we should now consider that the material parameters are functions of position vector, i.e., $\varepsilon(\mathbf{r})$ and $\mu(\mathbf{r})$. Taking this into account, we should now consider the monochromatic RS vector as

$$\mathbf{F}^\lambda(\mathbf{r}, t) = \frac{1}{\sqrt{2}}\left[\frac{\mathbf{D}(\mathbf{r})}{\sqrt{\varepsilon(\mathbf{r})}} + \lambda i\frac{\mathbf{B}(\mathbf{r})}{\sqrt{\mu(\mathbf{r})}}\right]e^{-i\omega t}. \quad \text{(C1)}$$

The expressions given in Eqs. (10) and (11) of the main text can be obtained in a few different ways. Probably, the simpler way is to compute the time derivative of the field defined in Eq. (C1) and, then, substitute the electric and magnetic fields as superpositions of $\mathbf{F}^+(\mathbf{r},t)$ and $\mathbf{F}^-(\mathbf{r},t)$. Let us indicate the important steps for the derivation. First, we should take the time derivative and substitute the Faraday-Ampére's equations:

$$\partial_t \mathbf{F}^\lambda(\mathbf{r},t) = \frac{1}{\sqrt{2}}\left[\frac{-i\omega\mathbf{D}(\mathbf{r})}{\sqrt{\varepsilon(\mathbf{r})}} + \lambda\frac{\omega\mathbf{B}(\mathbf{r})}{\sqrt{\mu(\mathbf{r})}}\right]e^{-i\omega t} = \frac{1}{\sqrt{2}}\left[\frac{\nabla\times\mathbf{H}(\mathbf{r})}{\sqrt{\varepsilon(\mathbf{r})}} - \lambda i\frac{\nabla\times\mathbf{E}(\mathbf{r})}{\sqrt{\mu(\mathbf{r})}}\right]e^{-i\omega t}. \quad \text{(C2)}$$

Then, we should substitute the electric and magnetic fields by the monochromatic RS vector defined in Eq. (C1):

$$\mathbf{E}(\mathbf{r}) = \frac{1}{\sqrt{2\varepsilon(\mathbf{r})}}[\mathbf{F}^+(\mathbf{r}) + \mathbf{F}^-(\mathbf{r})], \quad \text{(C3)}$$

$$\mathbf{H}(\mathbf{r}) = \frac{-i}{\sqrt{2\mu(\mathbf{r})}}[\mathbf{F}^+(\mathbf{r}) - \mathbf{F}^-(\mathbf{r})], \quad \text{(C4)}$$

where we have identified the spatial part of the monochromatic RS vector as $\mathbf{F}^\lambda(\mathbf{r}) = \mathbf{F}^\lambda(\mathbf{r},t)e^{i\omega t}$. Finally, the relations given in Eqs. (10) and (11) of the main text are obtained by applying the appropriate vector calculus identities.

APPENDIX D: CONSERVATION OF $\hat{\mathbf{P}}^2$ FOR QUANTUM NONRELATIVISTIC MASSIVE PARTICLES

In this Appendix, we show that the conservation of $\hat{\mathbf{P}}^2$ for quantum nonrelativistic massive particles does not correspond to the restoration of a symmetry, but just to the properties of particular scattering potentials.

In our previous discussion, we have focused on the description of electromagnetic waves in different media. However, the mathematical tool that has been employed, i.e., the symmetry-breaking principle applied to the Poincaré group, does not mind about the nature of the particle it is describing. In other words, the mathematical principle applies regardless of whether the physical theory represents a massless or massive particle [42]. In this line, we have shown that the identities given by Eqs. (4)–(6) can also represent the dynamic equations of a nonrelativistic massive particle, i.e., the time-independent Schrödinger's equation with a constant potential. This implies that the UIRs of $P_{3,1}$ and its subgroups also play a fundamental role in the determination of the wave functions of massive particles [47]. In the upcoming section, we focus on the role of the square of linear momentum operator: we seek a physical situation for massive particles in which $\hat{\mathbf{P}}^2$ is a conserved quantity, regardless of the geometry of the problem.

Such a situation would be reached by the scattering of a massive particle in an inhomogeneous potential, $\hat{V}(\mathbf{r})$, for which the solutions still remain being eigenstates of the $\hat{\mathbf{P}}^2$ operator, regardless of the symmetries of the problem. This would similarly imply that a Casimir operator of the $P_{3,1}$ group can also be preserved in inhomogeneous environments. Interestingly, there is a potential that fulfills the requirements above: the generalized quantum hard-sphere potential, which can be defined as

$$\hat{V}_{hs}(\mathbf{r}) = \begin{cases} \infty & \text{if} \quad \mathbf{r}\in U_i \\ V_0 & \text{else}, \end{cases} \quad \text{(D1)}$$

where U_i with $i = 1, 2, ..., N$ are N arbitrary regions of the three dimensional space and V_0 represents a finite and constant potential energy. Crucially, due to the arbitrariness of the U_i regions, such a potential can be constructed with any possible geometry. The solutions of the time-independent Schrödinger's equations in such a potential, $\psi_{hs}(\mathbf{r})$, are most generally determined by the following constraints:

$$\begin{cases} \psi_{hs}(\mathbf{r},t) = 0 & \text{if} \quad \mathbf{r}\in U_i \\ \hat{\mathbf{P}}^2\psi_{hs}(\mathbf{r},t) = 2m(E-V_0)\psi_{hs}(\mathbf{r},t) & \text{else}. \end{cases} \quad \text{(D2)}$$

As a result, it can be checked that the wave functions which are solutions to the generalized quantum hard-sphere potential fulfill $\hat{\mathbf{P}}^2\psi_{hs}(\mathbf{r},t) = p^2\psi_{hs}(\mathbf{r},t)$, $\forall\,\mathbf{r},t$ with $p = \sqrt{2m(E-V_0)}$.

From the properties above, it directly follows that the square of linear momentum is a conserved magnitude in the scattering with a hard sphere potential. Indeed, one can show that the expected value of $\hat{\mathbf{P}}^2$ remains constant for such a potential, i.e.,

$$\frac{d}{dt}\left(\int d\mathbf{r}\,\psi^*_{hs}(\mathbf{r},t)\big[\hat{\mathbf{P}}^2\psi_{hs}(\mathbf{r},t)\big]\right) = 0. \quad \text{(D3)}$$

This implies that if at an initial instant solutions are eigenstates of $\hat{\mathbf{P}}^2$, this property continues to hold in the course of time. In other words, one usually says that p is a good quantum number. Note that exactly the same holds for the kinetic energy, $\hat{T} = \hat{\mathbf{P}}^2/2m$, as the mass is a fixed parameter in nonrelativistic quantum mechanics. Both in quantum and classical scattering theory the conservation of kinetic energy is just invoked as a property of certain types of interactions. We thus conclude that the preservation of the square of linear momentum is not, in general, associated with the restoration of a space-time symmetry.

APPENDIX E: ANALYTICAL DESCRIPTION OF THE SCATTERING OF ELECTROMAGNETIC WAVES WITH MAGNETIC SPHERES

The results provided in Fig. 3 are based on the Mie coefficients computed for a homogeneous sphere (with permittivity, ε_1 and permeability, μ_1) surrounded in a homogeneous medium (with permittivity, ε and permeability, μ). Note that, instead of the usual material parameters, we employ the refractive indices ($n_1 = \sqrt{\varepsilon_1\mu_1}$ and $n = \sqrt{\varepsilon\mu}$) and impedances ($Z_1 = \sqrt{\mu_1/\varepsilon_1}$ and $Z = \sqrt{\mu/\varepsilon}$). The coefficients which determine the field scattered by this system are

$$a_j(m,s,\mathrm{x}) = \frac{\psi_j(m\mathrm{x})\psi'_j(\mathrm{x}) - s\psi_j(\mathrm{x})\psi'_j(m\mathrm{x})}{\psi_j(m\mathrm{x})\xi'_j(\mathrm{x}) - s\xi_j(\mathrm{x})\psi'_j(m\mathrm{x})}, \quad \text{(E1)}$$

$$b_j(m,s,\mathrm{x}) = \frac{s\psi_j(m\mathrm{x})\psi'_j(\mathrm{x}) - \psi_j(\mathrm{x})\psi'_j(m\mathrm{x})}{s\psi_j(m\mathrm{x})\xi'_j(\mathrm{x}) - \xi_j(\mathrm{x})\psi'_j(m\mathrm{x})}, \quad \text{(E2)}$$

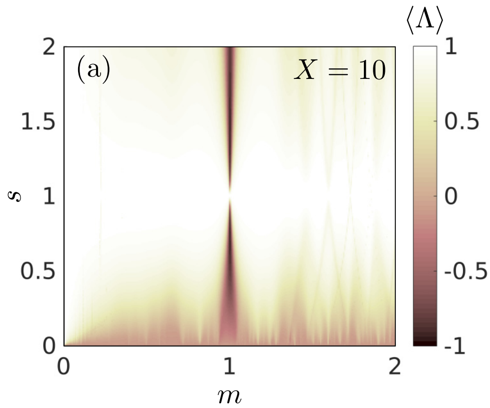

FIG. 4. Helicity expectation value for a magnetic sphere of size parameter $x = 10$ as a function of the refractive index contrast, m, and the impedance contrast, s.

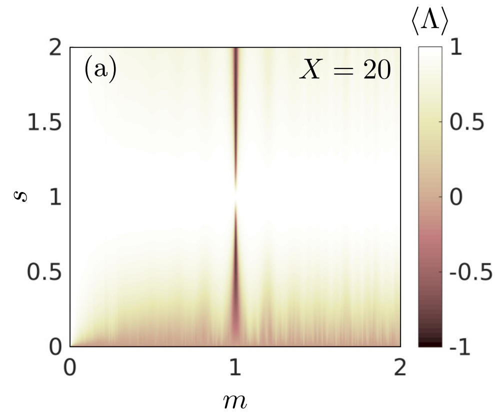

FIG. 5. Helicity expectation value for a magnetic sphere of size parameter $x = 20$ as a function of the refractive index contrast, m, and the impedance contrast, s.

where we have defined $m = n_1/n$ and $s = Z_1/Z$, and x is the size parameter. Moreover, $\psi_j(z)$ and $\xi_j(z)$ are the Ricatti-Bessel functions of order j. Finally, note that each of the scattering coefficients defined in Eqs. (E1) and (E2) are completely determined by three physical magnitudes: s, m, and x (see Ref. [60], Chap. IV). In addition, Fig. 3 displays the helicity expectation value, i.e., $\langle\Lambda\rangle$. This physical magnitude can be computed for any scattering problem and it is associated with the standard Stokes parameters I and V. In particular, it can be shown that the helicity expectation value can be computed as $\langle\Lambda\rangle = \int V d\Omega / \int I d\Omega$, where the integral takes over the whole solid angle [24]. Due to such a construction, it can be shown that $\langle\Lambda\rangle$ is a bounded magnitude, i.e., $\langle\Lambda\rangle \in [-1, +1]$, and that it indicates the degree of circular polarization of the scattered field components. In other words, it describes the helicity of the scattered electromagnetic field. On the other hand, the helicity expectation value (as many other magnitudes) depends on the incident illumination and, thus, to provide a general picture, in Fig. 3 we have opted for a standard circularly polarized plane wave as an illuminating field. In such a case, the helicity expectation value is given by the following closed expression:

$$\langle\Lambda\rangle = 2\frac{\sum_j (2j+1)\mathrm{Re}(a_j^* b_j)}{\sum_j (2j+1)\left(|a_j|^2 + |b_j|^2\right)}, \tag{E3}$$

where a_j and b_j are the scattering Mie coefficients defined in Eqs. (E1) and (E2). For different values of (m, s, x) parameters, this expression indicates whether helicity is preserved ($\langle\Lambda\rangle = +1$) or completely flipped ($\langle\Lambda\rangle = -1$) upon scattering. In Fig. 3 of the manuscript, we show how a sphere of a fixed size parameter ($x = 3$), when it is built from different refractive indices (m) and impedances (s), emits with different helicities. In particular, when fixing $s = 1$, it can be checked that helicity of the incident field is preserved, whereas for $m = 1$, the helicity is almost completely flipped.

This is a result of how the scattering coefficients relate under these particular conditions and can also be observed for spheres with different size parameters. In Fig. 4, we show that this behavior is also observed for spheres with a different size parameter, $x = 10$. On the other hand, in Fig. 5, we show that this behavior is also present when considering a sphere with even greater size parameter, $x = 20$.

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