

A general kinematic model for lubricated ball bearings based on the minimum energy hypothesis

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Abstract

Accurately predicting the performance of lubricated rolling elements is critical for certain key bearing applications. A widely employed simulation technique is quasi-static models, which although computationally efficient to implement, lack accuracy in predicting the kinematics of bearing components. This problem arises from their construction under kinematic assumptions, instead of solving the kinematic based on real frictional interactions. This paper presents a quasi-static model based on the minimum energy hypothesis for lubricated contacts, which determines the kinematic of the ball considering the effects of the frictional interactions governed by the elastohydrodynamic lubrication mechanism. The influence of kinematic variables on the sliding patterns and power loss distribution of the contact are analysed. The results are compared with other published quasi-static models, revealing substantial differences. The influence of the kinematic hypotheses on power losses are also studied at a bearing component scale. Finally, the effect of the governing lubrication mechanism of the contact is discussed, concluding that the limiting shear stress of the lubricant and the centrifugal forces have a significant effect on the kinematic of the ball.

Keywords: Ball bearing, Kinematics, Power loss, Elastohydrodynamic lubrication

1. Introduction

Ball bearings are widely used in rotary machine elements due to their capacity to operate under combined axial and radial loads at high operational speeds. In the majority of cases lubrication is employed to enhance performance and reliability. However, accurately describing the load and kinematics of the balls under lubricated conditions is challenging for three reasons. First, lubrication affects the friction of the contact which governs the rotation of the balls in the raceways, while the kinematics, in turn, directly influence friction [1]. Second, the ball-raceway contact is non-conformal, leading to high contact pressures. As a result, the behaviour of the properties of the lubricant under such conditions is difficult to determine [2]. Finally, the motion of the ball includes both rolling and spinning, which generates non-uniform sliding distribution within the contact [3]. These particularities present challenges in accurately predicting phenomena such as skidding [4], power losses [5, 6], stiffness [7], and fatigue life [8].

Two main theoretical formulations to describe the load and kinematic performance of the balls in bearings can be found in the scientific literature: quasi-static and dynamic models [9, 10]. Quasi-static models, employ static force and equilibrium equations for each ball. These equations consider various factors such as contact forces and inertial forces, allowing for the inclusion of dynamic effects [11]. In dynamic models, on the other hand, differential equations of motion are written for each ball [12, 13]. In these models, ball-raceway, ball-cage and raceway-cage interactions are considered, in addition to the dynamic effects. The system of equations is written in the time domain, and as a result transient phenomena can be predicted. The main disadvantage of dynamic models, however, is that they are complex to implement and require high computational resources [14].

Quasi-static models also present implementation challenges, the most critical of which involves incorporating realistic analytical friction expressions into the equilibrium equations [14]. The friction coefficient of the lubricated contacts depends on the load state and the kinematics of the ball, and in turn the motion of the ball is influenced by the friction coefficient. Hence, the problem needs to be solved simultaneously, taking into account these interconnected variables. However, convergence problems can arise as a result of the strong nonlinearity of these equations, and thus a common approach is to introduce kinematic constraints to simplify and solve the problem. It is therefore generally accepted in literature that quasi-static models can accurately predict load distribution, but they are not accurate in determining the kinematics of the bearing components [15, 16].

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Nomenclature

(O, X, Y, Z)	Bearing absolute coordinate system
(o', x', y', z')	Coordinate system of the ball
$(o_{i/o}, x_{i/o}, y_{i/o}, z_{i/o})$	Coordinate system of the contact
$\mathbf{A}_{i/o}^{cb}$	Transform matrix from the ball coordinate system to the contact coordinate system
A_1	Axial distance between between the outer and inner raceway groove curvature centres
A_2	Radial distance between between the outer and inner raceway groove curvature centres
a_H	Semi-major axis of the contact ellipse
b_H	Semi-minor axis of the contact ellipse
C	Ratio of the spinning friction moment
C_a	Axial clearance
D	Diameter of the ball
E	Young's modulus
d_m	Bearing pitch diameter
F_a	External axial force
F_c	Centrifugal force
F_r	External radial force
$F_{ri/o}$	Frictional force
J	Mass moment of inertia of the ball, $J = \frac{1}{60} \rho \pi D^5$
K	Load-deformation coefficient
k_e	Ellipticity ratio
M	External tilting moment
M_g	Gyroscopic moment
m	Mass of the ball, $m = \frac{1}{6} \rho \pi D^3$
P	Total power loss of the ball
$P_{i/o}$	Total power loss of the inner/ outer contact
P_{cage}	Cage power loss
P_{drag}	Drag power loss
P_r	Rolling power loss
P_s	Sliding power loss
p	Contact pressure
$p_{H,i/o}$	Inner/ outer maximum contact pressure
R_f	Radius of locus of the inner raceway groove curvature centres
$R_{i/o}$	Inner/ outer radius of curvature of the deformed contact surface
$r_{i/o}$	Inner/ outer raceway curvature radius
$r'_{i/o}$	Inner/ outer pure rolling radius
v_e	Entrainment speed
v_s	Sliding speed
v_b	Linear velocity of the ball in the reference system of the contact
v_r	Linear velocity of the raceway in the reference system of the contact
$W_{i,o}$	Inner/ outer normal load
Z	Number of balls
α^0	Nominal contact angle
$\alpha_{i/o}$	Inner/ outer contact angle
β	Pitch angle
δ_a	Displacement of the inner ring in the axial direction
δ_r	Displacement of the inner ring in the radial direction
$\delta_{i/o}$	Elastic deformation of the inner/ outer ball-raceway contact
$\Delta_{i/o}$	Distance between the inner/ outer raceway groove curvature centres
θ	Angular displacement of the inner ring
η	Low shear viscosity of the lubricant
η_G	Non-Newtonian viscosity of the lubricant
ν	Poisson's ratio

Λ	Limiting stress-pressure coefficient
Θ	Temperature of the fluid
ρ	Density
φ	Angular position of the ball (Azimuth angle)
ω	angular velocity of the shaft
ω_m	Orbital velocity of the ball
$\omega_{i/o}$	angular velocity of the inner/ outer ring
ω_R	angular velocity of the ball about its own axis
ω_{sp}	Spinning speed of the contact

One of the first and most commonly employed assumptions in the literature is the raceway control hypothesis presented by Jones [17, 18]. This hypothesis assumes that the ball rolls without spin in one raceway, and rolls and spins simultaneously in the opposite raceway. The raceway in which pure rolling motion occurs is defined as the controlling race, and is highly dependent on the operational conditions to which the bearing is subjected. The literature states that when bearings operate at low speeds, the inner control hypothesis provides accurate results. In contrast, when the bearing is working at high speeds, better results are obtained with the outer control [3, 11, 19, 20]. In this way, the behaviour of the raceway control transitions from low to high angular velocities. The experimental works of Kawamura [21], Chapman [22] and Brecher [23] confirmed the abovementioned transitional trend and concluded that the rheology of the lubricant has to be considered when predicting the bearing performance. However, the raceway control hypothesis does not consider the effect of rheology and hence cannot predict this transition. Although the accuracy of the raceway control hypothesis is not yet validated, it is widely employed to model bearing performance [24, 25, 26, 27, 28, 8, 29]. As an application example, Liu et al. [30] calculated and validated the bearing stiffness with high degree of accuracy, following the raceway control hypothesis.

An alternative approach to replace the raceway control hypothesis and be able to determine the kinematic of the ball was proposed by Harris [31, 32]. In this quasi-static model, the frictional interactions governed by the elastohydrodynamic lubrication (EHL) phenomena are written in terms of the kinematic variables and are introduced in equilibrium equations as unknown variables. As a result, the kinematic of the ball is calculated considering the effects of the frictional interactions, rather than following the kinematic constraints. The model showed good agreement with the experimental results of Poplawski and Mauriello [33], demonstrating the inadequacy of the raceway control assumption. Following a similar strategy, Gentle and Boness [34, 35] presented a quasi-static model based on a simplified EHL analytical equation. The authors introduced cage drag, pocket friction, and elastic deflection effects into the equilibrium equations, and validate the results with experimental measurements. However, incorporating the EHL expression in the equilibrium equations is proved challenging and the EHL expression needed to be simplified to such an extent that the effects of ball-raceway contacts on friction could not be accurately considered. For this reason, this strategy is more widely used in dry contact models, which are easier to predict than under lubricated conditions. Some examples include the work of Chen [36, 37], which considered Kalker's dry contact theory [38], the work developed by Servais et al. [39, 40, 41], Leblanc [42] and the recently published work of Zhao [43].

Another alternative to the raceway control hypothesis was developed by Ding [44]. Based on d'Alembert's inertia principle, this approach considers that the power conducted by all the forces on the ball is zero. Wang [20] extended this work, and by analysing the combined action of axial, radial, and moment loads in the kinematic variables, demonstrated that the outer raceway control assumption is appropriate for high operational speeds. The d'Alembert's hypothesis was also adopted by Gao [45], Jinhua [7], and Zhang [46]. The latter established a novel force triangulation method to efficiently predict load distribution. Although this approach has gained traction in the literature, however, there is no clear experimental evidence to confirm its accuracy. One further approach to replace the raceway control hypothesis is to calculate the pitch angle as the average value of the inner and outer contact angles. In the literature this model is known as the geometric hypothesis. Foord [47] experimentally measured the pitch angle of the ball and found that the pitch angle follows this trend. This approach has also been implemented by other authors such as Noel et al. [48] and Tong [49, 50, 51]. Nevertheless, these models are based on kinematic constraints and are not able to consider the influence of the lubricated contact interactions, when solving the kinematic of the ball.

To address the limitations of the above mentioned quasi-static models, several authors have proposed the minimum power dissipation concept to determine the kinematic of the ball. This hypothesis states that the ball will move in such a manner that the least amount of sliding energy is dissipated. However, this method have been developed for dry contact conditions [52, 53, 19, 54], or use a single constant value to determine the friction coefficient of the contact area [47], rendering this assumption invalid in the case of lubricated contacts.

More recently, Gupta [55] proposed a quasi-static model based on the minimum energy hypothesis for lubricated contacts, in which an EHL expression was introduced to describe the power loss of the contact [56]. The model exhibited good agreement with the reference dynamic model ADORE [14]. Nonetheless, it was constructed under the assumption that the friction coefficient and the power loss are only affected by the sliding speed component in the rolling direction. It fails to consider the sliding component in the transverse direction generated by the spinning motion, which contributes to shear stress distribution and the total power loss of the contact [57]. Furthermore, the work did not analyse the influence of working conditions such as angular velocity, axial load and the temperature of the lubricant on the kinematic of the contact. Each of these, affect the local rheological properties of the lubricant and thus have an impact on friction behaviour and kinematics.

This paper, therefore, presents an alternative quasi-static model for lubricated contacts based in the minimum power loss hypothesis. Power losses are calculated with a frictional EHL model, and the local effects produced by the spinning motion are considered. These innovations make it possible to solve the kinematic of the contact based on frictional interactions. The main objectives of the study are the following:

- (1) Develop a general theoretical quasi-static model of the ball-raceway contact based on the minimum energy hypothesis, using an EHL model to characterize the friction, and taking into account the local rheological effects produced by the particular kinematic of the interface.
- (2) Analyse the influence of the pitch angle and pure rolling line locations on the contact kinematics, and evaluate their effect on the power loss distribution of the contact.
- (3) Study the impact of the working conditions on the kinematic behaviour of the ball, and compare the results with those obtained from other kinematic hypotheses published in literature, such as the raceway control or D'Alembert's hypothesis.
- (4) Analyse the influence of the kinematics of the contact on the global power losses at the bearing component scale, by comparing the results obtained with other kinematic hypotheses.
- (5) Investigate the performance of the lubricant under a range of operational conditions and examine the influence of the governing lubrication mechanism on the power losses and kinematic behaviour of the ball.

2. Theoretical model development

2.1. Definition of the coordinate systems

The quasi-static bearing performance model was developed using three Cartesian coordinate systems. These are illustrated in Fig. 1, and their descriptions are as following:

1. Absolute coordinate system $[O, X, Y, Z]$: The origin O coincides with the centre of the axis. This coordinate system is fixed in space, while the X axis is collinear to the bearing axis of rotation.

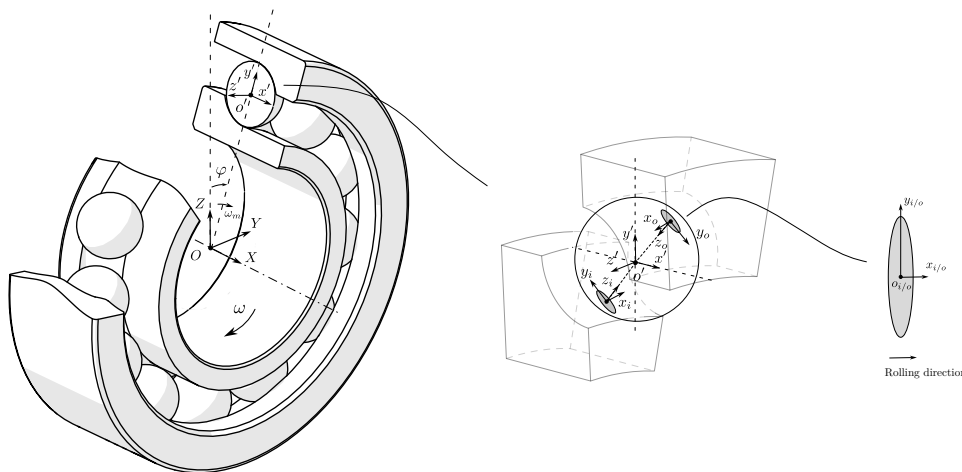


Figure 1: Reference systems of the ball bearing.

2. Coordinate system of the ball $[o', x', y', z']$: The origin o' of this coordinate system is located in the centre of each ball and the axis x' is parallel to the rotating axis of the bearing. The y' axis is collinear to the axis formed between the centre of the absolute system O and the centre of the coordinate system of the ball o' . Assuming that the shaft rotates with ω , the reference system of the ball rotates at an angular velocity ω_m with respect the absolute system and the position of the origin is defined with the variable φ .
3. Coordinate system of the contact $[o_i/o, x_i/o, y_i/o, z_i/o]$: The origin o of the system is fixed in the centre of the contact ellipse. The x axis is collinear to the semi-minor axis and the z axis is parallel to the normal direction of the contact surface, which passes through the centre of the ball. The positive direction of the z axis is oriented towards the centre of the ball. Each ball has two contact points and hence, two contact coordinate systems are defined for each ball. These are differentiated with the subscripts i and o , respectively.

2.2. Kinematic of the ball

Fig. 2 depicts the angular velocity vectors of the ball bearing system referenced in the coordinate system of the ball. The origin is fixed at the centre of the ball and it rotates at an orbital speed ω_m with respect to the absolute reference system. Therefore, both the inner and outer raceways are in relative motion with respect to the centre of the ball, with angular velocities of ω_i and ω_o , respectively. ω_R is the angular velocity of the ball with respect to its axis of rotation, the orientation of which is determined by the pitch angle β . As stated by Harris the velocities of the ball and the raceways are defined as [3]:

$$\omega_R = \omega \left(\frac{r'_o (\cos \beta \cos \alpha_o + \sin \beta \sin \alpha_o)}{d_m/2 + r'_o \cos \alpha_o} + \frac{r'_i (\cos \beta \cos \alpha_i + \sin \beta \sin \alpha_i)}{d_m/2 - r'_i \cos \alpha_i} \right)^{-1} \quad (1)$$

$$\omega_o = \omega \left(1 + \frac{r'_i (d_m/2 + r'_o \cos \alpha_o) (\cos \beta \cos \alpha_i + \sin \beta \sin \alpha_i)}{r'_o (d_m/2 - r'_i \cos \alpha_i) (\cos \beta \cos \alpha_o + \sin \beta \sin \alpha_o)} \right)^{-1} \quad (2)$$

$$\omega_i = \omega \left(1 + \frac{r'_o (d_m/2 - r'_i \cos \alpha_i) (\cos \beta \cos \alpha_o + \sin \beta \sin \alpha_o)}{r'_i (d_m/2 + r'_o \cos \alpha_o) (\cos \beta \cos \alpha_i + \sin \beta \sin \alpha_i)} \right)^{-1} \quad (3)$$

$$\omega_m = \omega - \omega_i \quad (4)$$

where r'_i and r'_o are the pure rolling radii of the inner and outer contact, α_i and α_o are the contact angles, d_m the pitch diameter, and ω is the angular velocity of the shaft. The vectorial notation of these components in the reference system of the ball is expressed as follows:

$$\omega_R^b = [-\omega_R \cos \beta \quad \omega_R \sin \beta \quad 0]^T \quad (5)$$

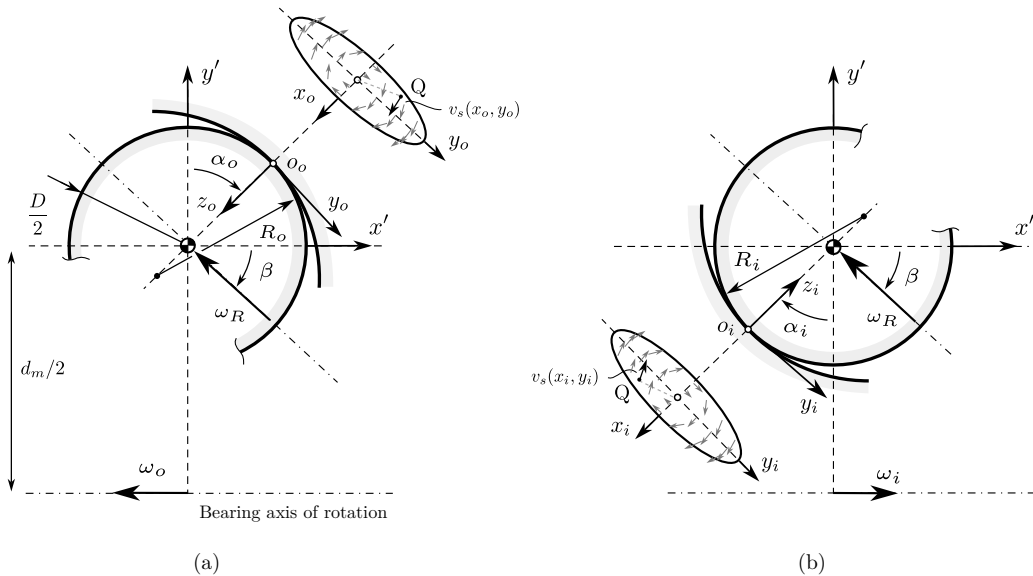


Figure 2: Kinematic of the ball and sliding patterns of the contact: (a) outer ball-raceway contact, and (b) inner ball-raceway contact.

$$\omega_i^b = [\omega_i \ 0 \ 0]^T \quad (6)$$

$$\omega_o^b = [-\omega_o \ 0 \ 0]^T \quad (7)$$

The combined effect of the curved contact surface and the spinning motion of the ball produces a non-uniform sliding speed distribution across the contact surface. Fig. 2 illustrates the sliding speed distribution of ball bearing contacts, and highlights an arbitrary point Q (x, y) of the contact. The sliding speed component of this point is calculated in the reference system of the contact, and is the difference between the linear velocity of the ball and the raceway.

$$\mathbf{v}_{s,k} = \mathbf{v}_{b,k} - \mathbf{v}_{r,k} \quad (k = i, o) \quad (8)$$

where $\mathbf{v}_b$ is the velocity of the ball, $\mathbf{v}_r$ the velocity of the raceway and the subscripts i and o refer to the inner and outer contact respectively. The linear velocity of the ball is calculated as:

$$\mathbf{v}_{b,k} = \omega_{R,k}^c \times \mathbf{s}_{b,k} \quad (k = i, o) \quad (9)$$

in which $\mathbf{s}_{b,k}$ is the position vector and $\omega_{R,k}^c$ the angular velocity of the ball. The angular velocity of the ball presented in Eq. (5) is written in the coordinate system of the ball (superscript b). Since the sliding speed distribution is analysed in the coordinate system of the contact (superscript c), the following vectorial transformation must be conducted:

$$\omega_{R,k}^c = \mathbf{A}_k^{cb} \omega_R^b \quad (k = i, o) \quad (10)$$

with $\mathbf{A}_k^{cb}$ representing the transformation matrix which transforms the reference system of the ball, into the reference system of the contact:

$$\mathbf{A}_k^{cb} = \begin{bmatrix} 0 & 0 & 1 \\ \cos \alpha_k & -\sin \alpha_k & 0 \\ \pm \sin \alpha_k & \pm \cos \alpha_k & 0 \end{bmatrix} \quad (k = i, o) \quad (11)$$

where the upper sign of the $\pm$ operand makes reference to the inner contact and the lower sign is applied to the outer contact. The signs used in Eq. (16) and Eq. (18), follow the same criteria.

The position vector between the rotating axis of the ball and any point Q (x, y) of the contact surface is defined as follows:

$$\mathbf{s}_{b,k} = \left\{ \begin{array}{c} x_k \\ y_k \\ - \left[(R_k^2 - y_k^2)^{1/2} - (R_k^2 - a_{H,k}^2)^{1/2} + ((D/2)^2 - a_{H,k}^2)^{1/2} \right] \end{array} \right\} \quad (k = i, o) \quad (12)$$

where, a_H is the semi-major axis of the contact ellipse and D is the diameter of the ball. R_k is the radius of curvature of the deformed pressure surface calculated as [3]:

$$R_k = \frac{2r_k D}{2r_k + D} \quad (k = i, o) \quad (13)$$

in which r_k is the raceway curvature radius.

Following the same procedure, the velocity of the raceway is defined as:

$$\mathbf{v}_{r,k} = \omega_k^c \times \mathbf{s}_{r,k} \quad (k = i, o) \quad (14)$$

The rotational vectors of the raceways are also transformed into the reference system of the contact:

$$\omega_k^c = \mathbf{A}_k^{cb} \omega_k^b \quad (k = i, o) \quad (15)$$

The position vectors of the inner and outer contact are determined as:

$$\mathbf{s}_{r,k} = \left\{ \begin{array}{c} x_k \\ y_k \\ \pm \left[\frac{d_m}{2 \cos \alpha_k} \mp \left((R_k^2 - y_k^2)^{1/2} - (R_k^2 - a_{H,k}^2)^{1/2} + ((D/2)^2 - a_{H,k}^2)^{1/2} \right) \right] \end{array} \right\} \quad (k = i, o) \quad (16)$$

The distance between the raceway groove curvature and the centre of the ball is:

$$\Delta_{k,j} = (r_k - 0.5 D) + \delta_{k,j} \quad (k = i, o) \quad (22)$$

where, $\delta_{k,j}$ is the normal contact deformation, which is related to the contact normal force in accordance with Hertz's law [3]:

$$W_{k,j} = K \delta_{k,j}^{3/2} \quad (k = i, o) \quad (23)$$

in which K represents the load-deformation coefficient. Finally, finding the geometrical compatibility in Fig. 3, the following two equations are obtained:

$$\Delta_{i,j} \sin \alpha_{i,j} + \Delta_{o,j} \sin \alpha_{o,j} - A_{1,j} = 0 \quad (24)$$

$$\Delta_{i,j} \cos \alpha_{i,j} + \Delta_{o,j} \cos \alpha_{o,j} - A_{2,j} = 0 \quad (25)$$

Fig. 4 depicts the forces and moments acting on a j^{th} ball of the bearing. The static equilibrium condition of the ball is defined by the following three equations:

$$\sum F_{x'} = 0 \rightarrow W_{i,j} \sin \alpha_{i,j} - W_{o,j} \sin \alpha_{o,j} + F_{ri,j} \cos \alpha_{i,j} - F_{ro,j} \cos \alpha_{o,j} = 0 \quad (26)$$

$$\sum F_{y'} = 0 \rightarrow W_{i,j} \cos \alpha_{i,j} - W_{o,j} \cos \alpha_{o,j} + F_{ri,j} \sin \alpha_{i,j} - F_{ro,j} \sin \alpha_{o,j} + F_{c,j} = 0 \quad (27)$$

$$\sum M_{z'} = 0 \rightarrow M_{g,j} - 0.5 D F_{ri,j} - 0.5 D F_{ro,j} = 0 \quad (28)$$

In these expressions, $M_{g,j}$ is the gyroscopic moment and $F_{c,j}$ the centrifugal force. These are derived from the orbital motion of the ball and calculated as follows:

$$M_{g,j} = J \omega_{R,j} \omega_{m,j} \sin \beta_j \quad (29)$$

$$F_{c,j} = 0.5 d_m m \omega_{m,j}^2 \quad (30)$$

where J is the mass moment of inertia along the z' axis, m the mass of the ball, ω_R is the angular velocity of the ball given in Eq. (1), and ω_m the orbital velocity of the ball defined in Eq. (4).

This model assumes that the motion of the ball is restricted in the z' axis by the frictional forces of the contact. The restriction is established following the approach of Wang [20], which assumes that the friction coefficient of the inner and outer contact is the same yielding the following expression:

$$\frac{F_{ri,j}}{W_{i,j}} = \frac{F_{ro,j}}{W_{o,j}} \quad (31)$$

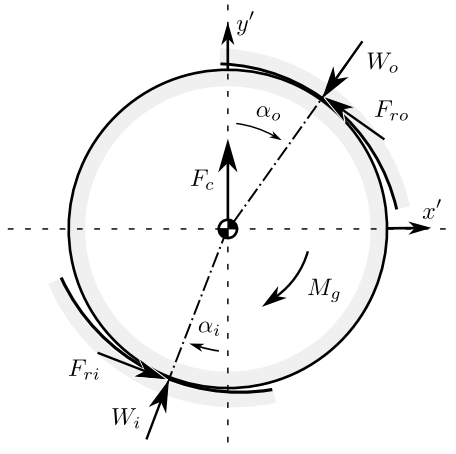


Figure 4: Free body diagram of the ball.

The load-deformation analysis presented above considers external displacements and for a single ball. The corresponding external forces and moments applied to the bearing are calculated by summing the reaction forces of all the Z balls:

$$F_a = \sum_{j=1}^Z (W_{i,j} \sin \alpha_{i,j} + F_{ri,j} \cos \alpha_{i,j}) \quad (32)$$

$$F_r = \sum_{j=1}^Z (W_{i,j} \cos \alpha_{i,j} - F_{ri,j} \sin \alpha_{i,j}) \cos \varphi_j \quad (33)$$

$$M = \sum_{j=1}^Z [(W_{i,j} \sin \alpha_{i,j} + F_{ri,j} \cos \alpha_{i,j}) R_f - F_{ri,j} r_i] \cos \varphi_j \quad (34)$$

where F_a represents the axial force, F_r the radial force and M the external tilting moment.

2.4. Sliding EHL power losses

The proposed model assumes that the ball will move in a manner that minimizes the total power loss of the ball. The total power loss of a lubricated ball bearing is calculated with Eq. (35) and it is derived from different sources, which are classified as load-dependent and load-independent [58, 59, 60]. The load dependent sources are the power losses generated by the rolling resistance (P_r) and those produced by the sliding resistance (P_s). The load-independent losses depend on the number of components of the bearing. The most common load-independent sources considered in the literature are the drag losses (P_{drag}) and power losses generated by the cage-ball interactions (P_{cage}).

$$P = \underbrace{\sum_{j=1}^Z \sum_{k=i,o} (P_{s,k} + P_{r,k})}_{\text{Load-dependent}} + \underbrace{[P_{drag} + P_{cage}]}_{\text{Load-independent}} \quad (35)$$

For specific working conditions, the kinematic of the ball is determined by β , r'_i and r'_o . Each of the power loss component is dominated by a different kinematic characteristic and therefore the variation of some of the kinematic variables do not influence the power losses.

- P_s : Sliding power loss is determined by the shear stress generated by the sliding speed distribution of the contact. β , r'_i and r'_o greatly influence the sliding speed distribution and hence the variation of these terms have a big effect in the sliding power loss.
- P_r : The rolling power loss is governed by the entrainment speed (v_e) and the contact load (W) of the ball [61]. The variation of β , r'_i and r'_o does not change the entrainment speed (v_e) nor the contact load (W) and hence the rolling losses are constant. Therefore, this term is not considered.
- P_{drag} : The power loss generated by the drag forces, are mainly influenced by the orbital speed (ω_m) of the balls [62]. However, the variation of β , r'_i and r'_o does not change the orbital velocity of the ball and hence the drag losses are constant. This term is not considered.
- P_{cage} : The proposed model does not consider the cage, and hence the power loss generated by this element is not considered in the model. It should be mentioned that, the cage and the relative lubrication could influence the local lubrication state of the contact as demonstrated by Liu et al.[63]. However, the proposed model assumes that the contact is well lubricated in every operational condition.

Since, the rolling, drag and cage power losses are not dependent on β , r'_i and r'_o , their consideration does not modify the value of the pitch angle at which the minimum power loss occurs. Hence, in order to simplify the calculation procedure, they are not considered in the model. The only power loss component that is taken into account is that generated by the sliding speed distribution. Applying the above mentioned assumptions to the Eq. (35), the total power loss of the j^{th} ball is calculated:

$$P = \sum_{k=i,o} P_{s,k} \quad (36)$$

The sliding power loss is determined by integrating the product of local sliding speed and tangential shear stress on the contact area. The spinning motion generates sliding speed components ($\mathbf{v}_{s,k}$) in different directions across the contact surface, and the consequent shear stress component (τ_k) is in the opposite direction. However, the total power loss is independent of vector directions, and hence the absolute values are used.

In lubricated contacts, the shear resistance is determined by the EHL mechanism. Under such conditions, the mating bodies are separated by a thin film, and hence the shear stress is governed by the viscous shearing of the lubricant. The rheological properties depend on the shear rate, pressure, and the temperature to which the lubricant is subjected. Such properties vary across the contact interface, and problem is therefore solved with a localised analysis. In this regard, the contact domain is discretised into a mesh of 512x512 nodes which ensures a favorable compromise between the computational effort and the accuracy of the results.

The theoretical EHL model employed in the present work is an adaptation of the model proposed by Iribecampes et al. [57]. The contact is considered to be in the full film lubrication regime, meaning that there is no interaction between the asperities of the surfaces. The film thickness (h_c) is considered constant and is calculated with the central film thickness formula proposed by Chittenden [64]. The pressure (p) follows the Hertzian dry contact distribution [65]. In this model, isothermal temperature distribution is assumed at the contact interface. Based on these assumptions, the power loss of the lubricated contact is calculated as:

$$P_{s,k} = \iint_{A_H} |\mathbf{v}_{s,k}| |\tau_k| dA_H = \iint_{A_H} |\mathbf{v}_{s,k}| |\min(\dot{\gamma} \eta_G, \Lambda p)| dA_H \quad (k = i, o) \quad (37)$$

where $\dot{\gamma}$ is the shear rate of the contact, defined as $\dot{\gamma} = |\mathbf{v}_{s,k}| / h_c$, η_G is the generalised viscosity of the lubricant and Λ is the limiting stress-pressure coefficient.

The generalised viscosity of the lubricant is modeled assuming non-Newtonian behaviour $\eta_G = f(\dot{\gamma}, p, \Theta)$. Under certain operating conditions, the shear rate of some points of the contact is significant and hence shear thinning effects must be considered. This behaviour is characterised following the shifted Carreau model [66]:

$$\eta_G(\dot{\gamma}, p, \Theta) = \eta \left[1 + \left(\dot{\gamma} \lambda_R \frac{\eta}{\eta_R} \frac{\Theta_R}{\Theta} \frac{V}{V_R} \right)^2 \right]^{(n-1)/2} \quad (38)$$

In this equation, η is the low shear rate viscosity of the lubricant and V is the volume of the fluid. The temperature and pressure dependence of the Newtonian viscosity is calculated with the improved Yasutomi model [67]:

$$\eta(p, \Theta) = \eta_g \exp \left[\frac{-2.303 C_1 (\Theta - \Theta_g) F}{C_2 + (\Theta - \Theta_g) F} \right] \quad (39)$$

$$\Theta_g = \Theta_{g0} + A_1 \ln(1 + A_2 p) \quad (40)$$

$$F = (1 + b_1 p)^{b_2} \quad (41)$$

The influence of the contact pressure and temperature on the volume is characterised following the Murganhan's equation of state [68]:

$$\frac{V}{V_R}(p, \Theta) = (1 + a_v (\Theta - \Theta_R)) \left(1 + \frac{K'_0}{K_0} p \right)^{\left(-\frac{1}{K'_0} \right)} \quad (42)$$

$$K_0 = K_{00} \exp(-\beta_K \Theta) \quad (43)$$

All the constant parameters of Eqs. (38)-(43) depend on the specific lubricant in use.

2.5. Solution procedure

Once the governing equations are presented, the procedure to solve the proposed model is introduced. The solution consists of calculating the unknown variables β , r'_i , r'_o , α_i , α_o , W_i and W_o from the equations presented above. The flowchart of the calculation process is illustrated in Fig. 5. The proposed model determines the unknown variables based on the minimum energy principle. The solution consists of iteratively finding the unknown variables which minimizes the sliding power loss of the contact. The process is divided into 6 steps:

Step 1 Introduce the input variables. These are classified in terms of geometry and material parameters of the bearing, lubricant variables and working conditions (δ_a , δ_r , θ , ω , Θ).

- Step 2** Initialise the unknown parameters. The initial values of the inner and outer pure rolling radii (r'_i, r'_o) and the pitch angle (β) are then established. The initial condition for the pure rolling radii is the radius of the ball, while the pitch angle is initialised assuming the nominal contact angle of the bearing.
- Step 3** Calculate the contact forces and contact angles. The geometry compatibility equations Eqs. (24)-(25), and the equilibrium equations Eqs. (26)-(28) define a system of five non-linear algebraic equations. The unknown variables of the system of equations are W_i , W_o , α_i , α_o and μ , which are numerically calculated following the Newton-Raphson method, with the initial values of 100 N for the contact loads, 40° for the contact angles and 0.03 for the friction coefficient.
- Step 4** Calculate the kinematic of the contact. First, the absolute angular velocities of the balls and the raceways are calculated with Eqs. (1)-(4). As the contact load is determined in the previous step, the semi-major (a_H) and semi-minor axes (b_H) of the contact ellipse can be defined and the sliding distribution is calculated with Eq. (8). Finally, the entrainment speed is calculated by Eq. (17).
- Step 5** Calculate the power loss of the contact. Determine the shear stress at each point of the contact surface and determine the power loss of the ball with Eqs. (36)-(43).
- Step 6** Find values for r'_i , r'_o and β that minimizes the sliding power loss of the ball. The process is a multivariable optimization problem which is conducted following Newton's gradient descent method presented in Eq. (44). The partial differential terms are calculated following the central difference scheme [69]. Steps 3-5 are repeated until the error (ε), (the difference between the k iteration and the $k + 1$ approximation) is below 0.1%.

$$\begin{bmatrix} r'_i \\ r'_o \\ \beta \end{bmatrix}_{k+1} = \begin{bmatrix} r'_i \\ r'_o \\ \beta \end{bmatrix}_k - \underbrace{\begin{bmatrix} \frac{\partial P}{\partial r'_i} \left(\frac{\partial^2 P}{\partial r'^2_i} \right)^{-1} \\ \frac{\partial P}{\partial r'_o} \left(\frac{\partial^2 P}{\partial r'^2_o} \right)^{-1} \\ \frac{\partial P}{\partial \beta} \left(\frac{\partial^2 P}{\partial \beta^2} \right)^{-1} \end{bmatrix}_k}_{-\varepsilon} \quad (44)$$

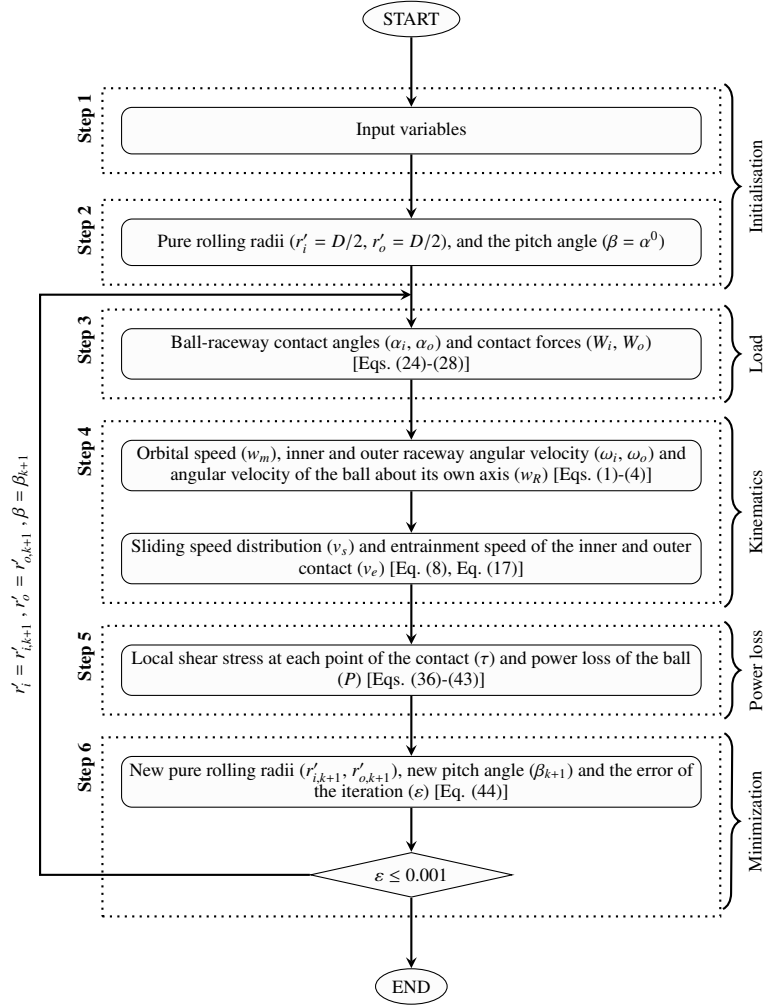


Figure 5: Flowchart of the minimum energy method.

3. Definition of the case studies

In the present research the proposed model was used to analyse the kinematic and power loss behaviour of a commonly used type of angular contact ball bearing (ACBB 7205). The main geometrical and material parameters of the bearing are listed in Table 1. The basic static load rating of the bearing is approximately 11 kN and the limiting speed is 26000 rpm [70]. These parameters were used as criteria to select the operating conditions for the case studies.

Table 1: Characteristics of the ACBB 7205.

Parameter	Value
Ball diameter, D [mm]	7.94
Pitch diameter, d_m [mm]	38.5
Raceway groove radius, $r_{i/o}$ [mm]	4.2
Ball-raceway nominal contact angle, α^0 [°]	40
Number of balls, Z [-]	13
Axial clearance, C_a [mm]	0
Material	AISI 52100
Young's modulus, E [GPa]	210
Poisson's ratio, ν [-]	0.3
Density, ρ [kg/m³]	7850

The lubricant selected for the study was squalane, which is representative of the rheological behaviour of low

viscosity paraffinic mineral oils or polyalphaolefins (PAO) used in bearing applications [68]. This lubricant is a commercially available low molecular weight branched alkane (2,6,10,15,19,23- hexamethyltetracosane) and is frequently used in theoretical and experimental EHL studies [71, 72]. The rheological and thermophysical properties are well-characterised in literature from high-stress viscometer measurements and non-equilibrium molecular dynamic simulations [66, 67, 73]. The constant parameters of the theoretical models followed to predict the rheological behaviour are summarised in Table 2.

Table 2: Rheological properties of squalane.

Property	Parameter
Non-Newtonian viscosity: Carreau's model [66]	$\lambda_R = 2.26 \cdot 10^{-9} \text{ s}$ $\Theta_R = 40^\circ\text{C}$ $\eta_R = 15.6 \cdot 10^3 \text{ Pas}$ $n = 0.463$
Low shear rate viscosity: Yasutomi model [67]	$\eta_g = 1.23 \cdot 10^7 \text{ Pas}$ $\Theta_{g0} = -88.69^\circ\text{C}$ $A_1 = 263.8^\circ\text{C}$ $A_2 = 0.3527 \text{ GPa}^{-1}$ $b_1 = 13.73 \text{ GPa}^{-1}$ $b_2 = -0.3426$ $C_1 = 11.66$ $C_2 = 39.17^\circ\text{C}$
Density: Murnaghan's EOS [68]	$a_v = 0.000835 \text{ K}^{-1}$ $K'_0 = 10.85$ $K_{00} = 8.824 \text{ GPa}$ $\beta_K = 0.006321 \text{ K}^{-1}$
Limiting stress-pressure coefficient [66]	$\Lambda = 0.075$

Two types of analyses were defined to compare the results obtained from the proposed model with other published models. The mathematical expressions to calculate the pitch angle and the solution procedure followed by different literature models are summarised in Appendix A. Although the proposed model was generalised to be able to calculate combined radial and axial loads, only axial loads are considered in the present study. As a result, the balls are uniformly loaded. The details of the two case of studies are summarised in Table. 3.

- Case 1: Aims to understand the effect of the pitch angle on the kinematic and the power loss of the contact, introducing the fundamental principle of the minimum energy hypothesis. In this regard, the simulations were conducted for a determined range of prescribed pitch angle values, and the power loss generated at each angle was analysed. Specifically, the pitch angle was evaluated every 0.5° , between 30° and 60° , and each was simulated under the same operational condition ($F_a = 1500 \text{ N}$, $\omega = 5000 \text{ rpm}$, and $\Theta = 40^\circ\text{C}$). The evolution of the sliding speed distribution and the consequent power loss distribution of the contact was analysed for each pitch angle orientation.
- Case 2: Aims to determine the influence of the operational conditions on the pitch angle of the ball. For that purpose, the pitch angle was calculated for a wide range of angular velocities, axial loads and temperatures, and their individual effects were analysed. As an example of industrial applications, the combination of high angular velocity and low loads employed in this case of study could be aligned with turbine rotors. The combination of lower speeds and high loads could be related to industrial gearboxes, while the combination of high speeds and moderate loads could be associated with spindles. The results were compared with other models frequently used in the literature. The effect of different kinematic hypotheses in the power loss of the bearing component was also analysed.

Table 3: Simulated conditions for each case study.

Parameter	Case 1	Case 2
Axial load, F_a [N]	1500	1500 - 9000
Inner pressure, p_{Hi} [GPa]	1.56	1.56 - 2.76
Outer pressure, p_{Ho} [GPa]	1.30	1.30 - 2.33
angular velocity, ω [rpm]	5000	200 - 30000
Temperature, Θ [°C]	40	20 - 100
Bearing type	ACBB 7205	
Lubricant	Squalane	

4. Results

4.1. Influence of the pitch angle on the power loss and the sliding speed distribution of the contacts

Fig. 6 depicts the evolution of the power loss in a ball as a function of the pitch angle, for a single operating condition. The power losses of the inner and outer contacts are also illustrated to show their individual contributions.

The minimum power loss of the outer contact occurs when its spinning speed is zero. Under this condition, the ball rolls without spin along the outer contact and simultaneously rolls and spins along the inner contact. This state is in line with the outer raceway control (ORC) hypothesis, which is frequently used in the literature. Similarly, the minimum power loss of the inner contact occurs when its spinning speed is zero. In this case, the inner contact only involves a rolling motion, while both rolling and spinning motions occur along the outer contact. This condition is in agreement with the inner raceway control (IRC) hypothesis. It should be noted that these two conditions indicate the limiting rolling-spinning state that the contacts of the ball can be subjected to.

The proposed model (highlighted with a red dot in Fig. 6) assumes that the ball will move in such a way that minimizes power dissipation. This condition occurs when both contacts exhibit simultaneous spinning and rolling motions. In these specific working conditions, the pitch angle is closer to the IRC state, due to the relatively low angular velocity. The influence of the working conditions are analysed in detail in subsequent sections.

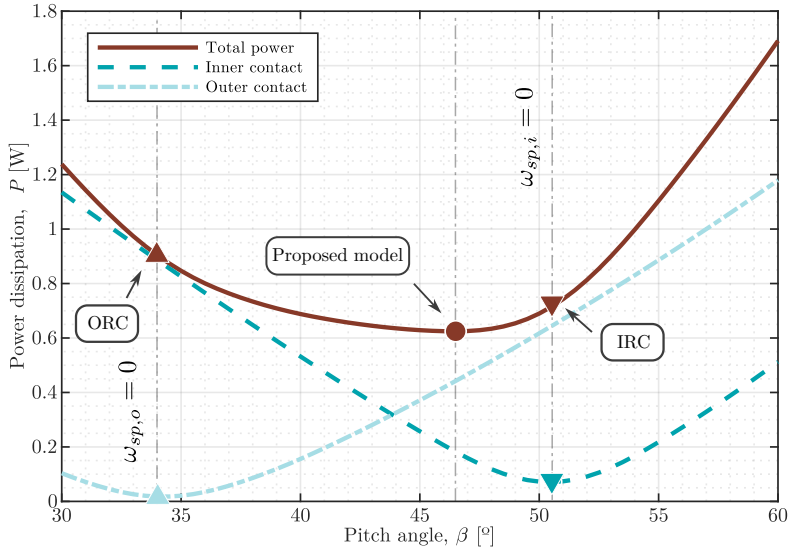
Figure 6: Power loss of the ball as a function of the pitch angle ($F_a = 1500$ N, $\omega = 5000$ rpm, and $\Theta = 40^\circ$ C).

Fig. 7 illustrates the effect of the pitch angle on the contact kinematics for inner and outer raceways of a bearing system. Note that the convexity of the inner and outer raceways is different, leading to a distinct ellipticity ratio at each contact. As a consequence, the same reaction force results in higher pressure of the inner contact.

Focusing on the kinematic state, the figure illustrates the sliding speed distribution of the contact for various pitch angle values. It can be observed that the pitch angle governs the spinning speed of the contact and hence the sliding speed distribution. Considering the pitch angle at which no spinning motion occurs as a reference state, when the pitch angle deviates from this reference condition, the spinning speed and sliding speed vectors are increased.

The black dots depicted in each sliding pattern determine the position of the spin pole (also known as the pure rolling line), which refers to the location of the contact ellipse where no sliding condition occurs. The location of the spin pole also changes with respect to the pitch angle. When the spinning speed is near zero, two spin poles can be observed in the contact ellipse. In the state where zero spinning occurs, these spin poles are symmetrical to the semi-minor axis. At higher spinning speeds, one spin pole appears within the contact and is always located close to centre of the ellipse. From a theoretical point of view, the optimal state that minimizes the power loss of a ball would be the condition under which simultaneous zero spinning occurs in both contacts. However, each pitch angle orientation defines a predetermined spinning speed for both inner and outer contacts. Therefore, it is not possible to obtain the simultaneous zero spinning state in both contacts.

Focusing on the ORC, the figure shows that the spinning speed of the outer contact is zero, and hence all the sliding vectors are aligned in the rolling direction. In this case, two spin poles are in the contact interface, meaning that no sliding condition occurs at these two points. The inner contact involves a simultaneous rolling and spinning motion, and as a consequence the sliding speed vectors form a circular pattern and the spin pole is located at the centre of the ellipse.

Regarding the sliding distribution predicted by the IRC hypothesis, the kinematic conditions are similar to ORC, but the rolling-spinning combination occurring along each contact is just the opposite. Therefore, in this case, the sliding speed vectors are aligned along the inner contact while the circular distribution occurs in the outer contact.

In contrast to the ORC and IRC, the sliding speed distribution obtained with the proposed model involves a simultaneous rolling and spinning motion in both contacts. This occurs when the spinning speed of the inner contact is lower than the outer contact. In addition, as the inner contact approaches the pure rolling condition, two spin poles appear in the contact, however, these are not symmetrical to the semi-minor axis of the ellipse.

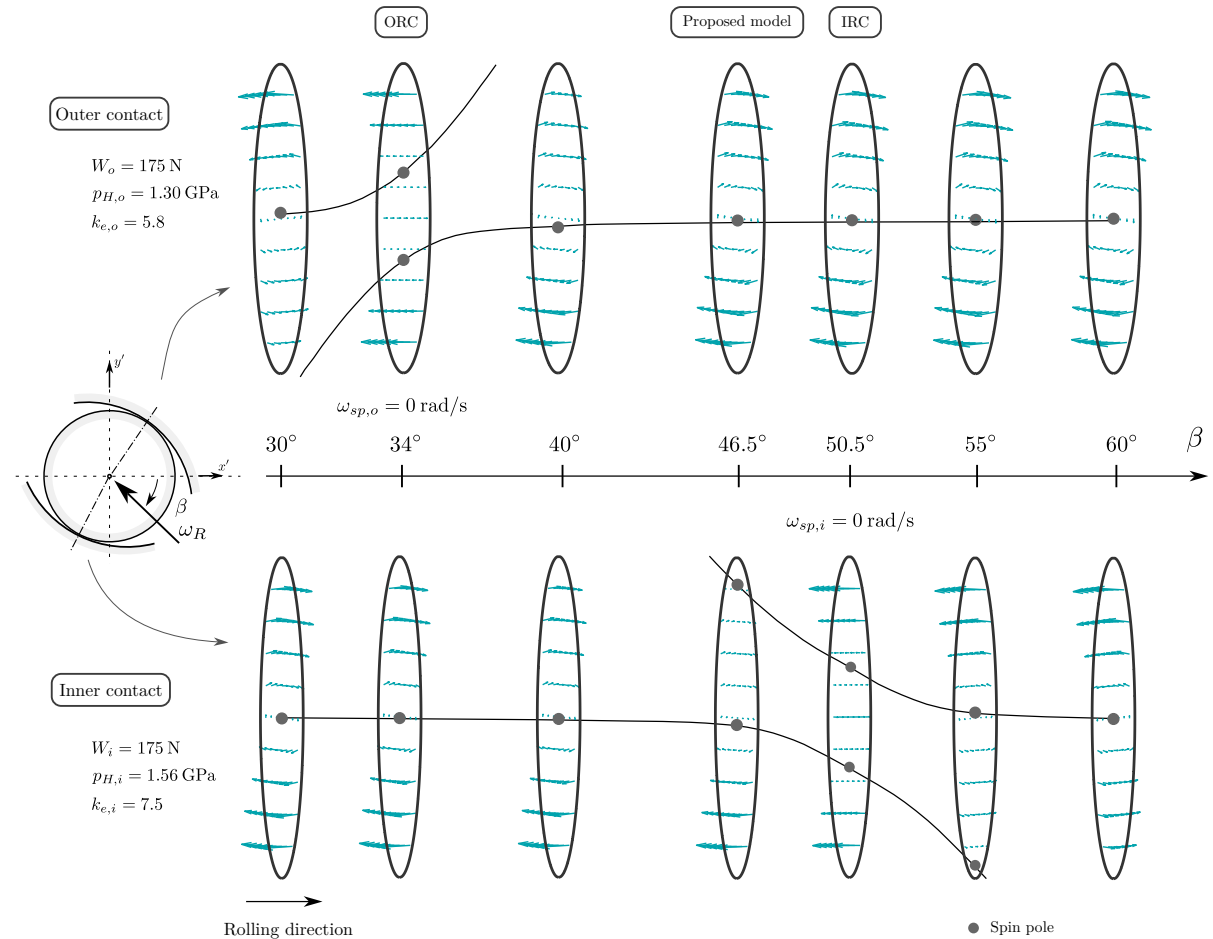


Figure 7: Sliding speed distribution of the contact for different pitch angle orientations ($F_a = 1500$ N, $\omega = 5000$ rpm and $\Theta = 40^\circ$ C).

Fig. 8 illustrates the power loss distribution of a ball-raceway contact for different pitch angle values. As previously stated, the pitch angle governs the sliding speed distribution, and hence it also has a substantial effect on the power loss distribution of the contact. This can be divided into three regimes:

- On the periphery of the contact interface, the Hertzian pressure tends to decrease. Consequently, the shear stress in this contact region is reduced, leading to a power loss that approaches zero.
- In the location of the spin pole, the sliding speed and the power losses are zero.
- In the rest of the contact area the power loss is governed by the kinematic of the contact. When the spinning speed is low, the local power loss of the contact is also low. Under such conditions, the distribution of the power loss tends to be homogeneous over the whole contact interface. At high spinning speeds, the local power loss is increased and in the lateral regions of the contact some peaks can be observed. The magnitude of these peaks increases with the spinning speed.

It should be noted that the distribution of these peaks is not symmetrical to the semi-minor axis of the ellipse. On one side of the ellipse their magnitude is slightly higher than on the other. The combination of the rolling and spinning speed components generates a non-uniform sliding distribution along the semi-major axis, leading to higher sliding speeds on one side of the ellipse. As a result, the peaks on that side are also higher.

Notably, when the spinning speed is zero, the power loss of the contact is not zero. This is because there exist sliding speed vectors aligned in the rolling direction (Fig. 7). These sliding components are generated by the elastic deformation of the ball and the raceway. Under this kinematic condition, the sliding distribution of the inner and outer contacts is the same, but the power loss of the inner contact is higher than the outer contact. This variation arises from the difference in contact pressure.

The rolling-spinning state of the inner and outer contact generated with each orientation of the pitch angle shows substantial differences in the sliding speed distribution of the contact, generating significant variations in

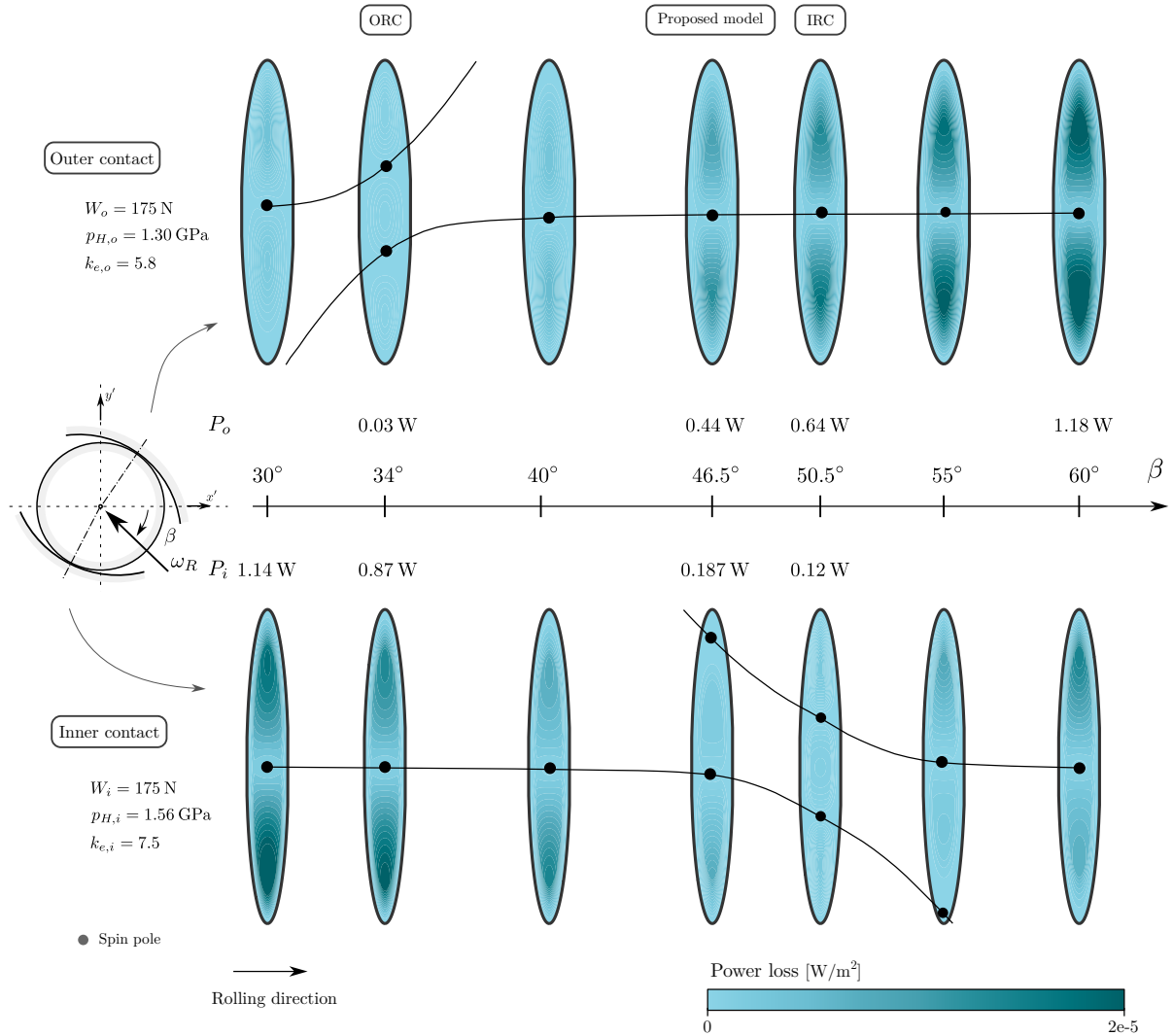


Figure 8: Power loss distribution of the contact for different pitch angle orientations ($F_a = 1500 \text{ N}$, $\omega = 5000 \text{ rpm}$, and $\Theta = 40^\circ \text{ C}$).

the power loss distribution. Therefore, the pitch angle is the main variable that governs the kinematic and power loss of the contact. Depending on the operating conditions, the optimal rolling-spinning state that minimizes the power loss of the contact will occur for a different orientation on the pitch angle.

4.2. Influence of the operational conditions on the pitch angle

Fig. 9 depicts the effect of the operating conditions on the pitch angle and compares the proposed model to the ORC and IRC hypotheses.

The influence of the axial load is shown in Fig. 9a. When the angular velocity is below 10000 rpm, the axial load has little influence on the pitch angle. This behaviour is the same for the IRC, ORC, and the proposed model. However, the pitch angle calculated with each method yields very different results. Under low speed conditions, the predictions of the proposed model are closer to the results of the IRC hypothesis, as discussed in the previous section.

In contrast, when the angular velocity is increased, both the axial load and the angular velocity begin to exert a stronger effect on the pitch angle. When the angular velocity is increased, the pitch angle calculated with the IRC hypothesis tends to increase, while that calculated with the ORC decreases. This effect is more pronounced when the axial loads are low. In the case of the proposed model, the pitch angle decreases when the angular velocity is increased, which is in line with the trend of the ORC hypothesis.

Fig. 9b shows the influence of the contact temperature on the pitch angle. To highlight the temperature effect, the results were analysed at 1500 N. Under this load the influence of the angular velocity on the pitch angle is higher, and hence the effect of the temperature is easier to observe. It is worth mentioning that the IRC and ORC hypotheses do not consider the effect of temperature when calculating the pitch angle. Consequently, the plots of these models remain constant regardless of temperature changes. On the other hand, the proposed model, is able to predict the effect of temperature. At low speeds the impact of the lubricant temperature is minimal and the pitch angle remains almost constant until the angular velocity is close to 10000 rpm. When the angular velocity is increased, the pitch angle tends to decrease, and this effect more pronounced at low contact temperatures.

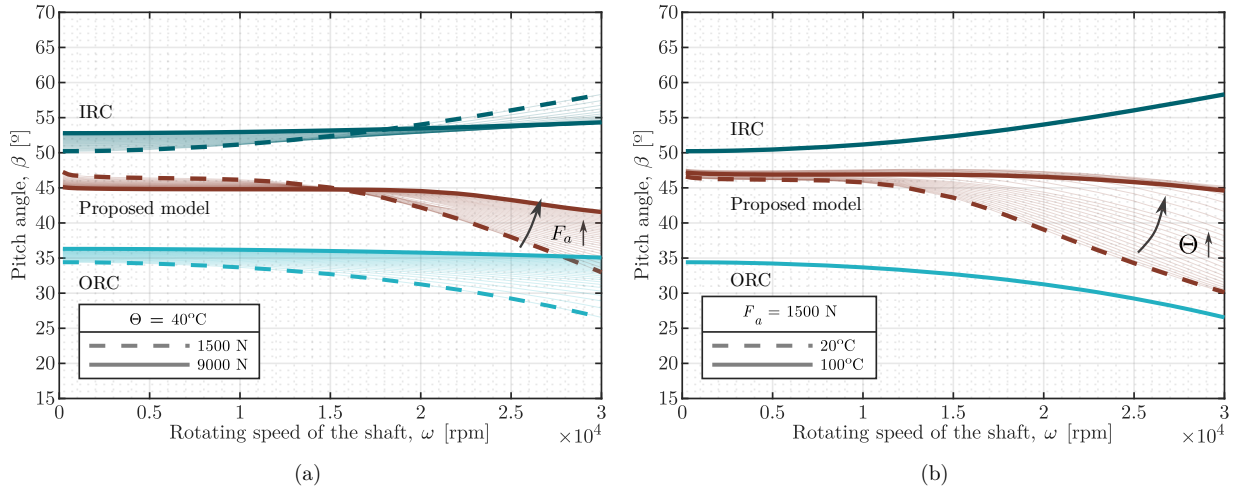


Figure 9: Effect of the working conditions on the pitch angle of the ball. (a) Influence of axial load and angular velocity on the pitch angle, and (b) influence of temperature and angular velocity on the pitch angle.

Fig. 9 illustrates the considerable difference between the pitch angle predicted with each raceway control hypothesis. In general, the scientific literature assumes a transitional trend of the raceway control from low to high angular velocities. For this reason, the IRC is considered to yield accurate results at low angular velocities, while the ORC makes better predictions at high angular velocities [3, 11, 19, 20]. To address this problem, some authors have proposed alternative hypotheses to replace the raceway control assumption, such as the geometric and D'Alembert's hypotheses.

To better understand this transitional behaviour, the results of these four hypotheses were compared with the proposed model (Fig. 10). The figure is plotted with a new variable $\beta\%$, which compares the results of the D'Alembert's (DAL) and geometric (GEO) hypotheses, and the proposed model to each raceway control hypothesis:

$$\beta\% = 100 \frac{(\beta - \beta_{ORC})}{(\beta_{IRC} - \beta_{ORC})} \quad (45)$$

where β_{IRC} is the pitch angle calculated following the IRC hypothesis and β_{ORC} is that predicted by the ORC hypothesis. When $\beta_{\%}$ is 0%, the pitch angle coincides with the ORC hypothesis, while a value 100% means that the pitch angle is governed by the IRC hypothesis.

The predictions obtained with the D'Alembert's hypothesis are closer to the ORC. The influence of the axial load is almost negligible, while an increase of the angular velocity causes a slight upward trend. In the case of the geometric hypothesis, $\beta_{\%}$ remains almost constant for all the simulated conditions. In both hypotheses, the effect of the operating temperature is not observed. This is because these hypotheses do not consider the frictional interactions of the contact, and hence the effects related to the lubricant cannot be predicted.

The results from the proposed model exhibit a transitional trend from the IRC to the ORC as the angular velocity increases. In terms of the influence of the axial load, as illustrated in Fig. 10a, the transitional behavior is more pronounced at lower loads. At high axial loads and low angular velocities, $\beta_{\%}$ exhibits minimal changes and remains constant at the middle of the IRC and ORC hypothesis. However, when the angular velocities are increased, $\beta_{\%}$ decreases and approaches the ORC.

Note that at 19000 rpm all the simulations conducted under every axial load coincide at approximately 50% of $\beta_{\%}$. This state of $\beta_{\%}$ exhibits a particular characteristic in which both the inner and outer contacts have equal spinning speed. This point occurs at different angular velocities depending on the contact temperature, which indicates that the kinematic of the ball is completely controlled by the rheology of the lubricant. Therefore, from a theoretical point of view, it would be possible to achieve a determined kinematic condition depending on the lubricant employed.

As regards the influence of the temperature of the lubricant, Fig. 10b depicts a similar transitional behaviour. As the angular velocity increases, the value of $\beta_{\%}$ decreases, and this effect is more pronounced at lower temperatures. At elevated temperatures, the decreasing trend is less significant, and $\beta_{\%}$ is closer to the IRC. This stabilisation occurs at high values of $\beta_{\%}$, indicating that the model approaches the IRC.

In summary, at low angular velocities the results of the proposed model align with the IRC, while at higher speeds the predictions tend toward the ORC. This, numerically confirms the general transitional behaviour found in the literature. Although the experimental results could not be replicated due to a lack of testing data, the observed transition is also similar to the experimental measurements conducted in the works of Chapman [22] and Brecher [23]. This confirms the accuracy of the proposed model.

On the other hand, the analysis shows that the D'Alembert's and geometric hypotheses are not able to predict the transitional trend from IRC to ORC. However, these hypotheses are able to calculate the orientation of the pitch angle which predicts a spinning motion in both contacts. This state closely approximates the real conditions of the ball.

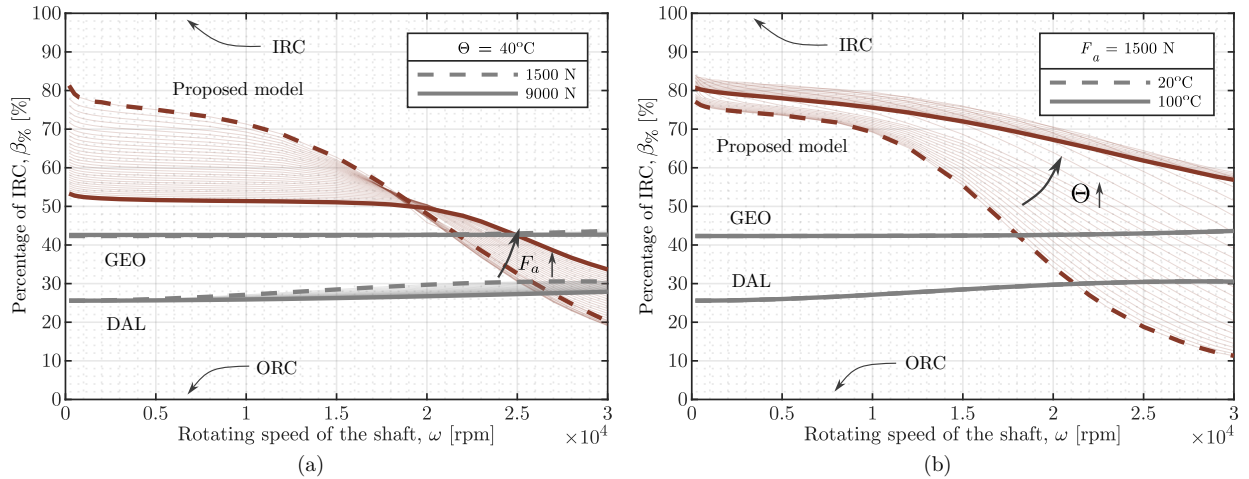


Figure 10: Effect of the working conditions in $\beta_{\%}$. (a) Influence of the axial load and the rotational speed, and (b) influence of the temperature and the rotational speed.

4.3. Effect of the bearing kinematics on the total bearing power loss

The results thus far have demonstrated that at the single ball level, the power loss of the contact is governed by the pitch angle. It has also been shown that the orientation of the pitch angle varies significantly depending on

the kinematic hypothesis followed. The next step is to analyse the effect of these predictions, but at the scale of the bearing component.

The total power loss of the component is the sum of the individual contributions of each ball. In the present case study, the ball is only loaded axially, and thus the contribution of each ball to the total system is equal. Fig. 11 shows the total power loss of the complete bearing system, together with the power loss of the inner and outer raceways, under a range of working conditions.

The figure shows that the geometric and D'Alembert's hypotheses achieve total power loss predictions similar to those of the proposed model under almost all the operational conditions. The geometric hypothesis predictions are closest to the proposed model.

However, higher differences can be observed in the comparison of the individual predictions of the inner and outer raceways. The highest deviations are those of the IRC and ORC hypotheses, under the lowest load condition (1500 N). These methods assume that all the spinning takes place in one raceway and hence most of the power loss occurs in that raceway. This approach overestimates the power loss.

In the geometric and D'Alembert's hypotheses the power loss is more uniformly distributed, and the behaviour approximates that predicted by the proposed model. Nonetheless, one major difference exists. The geometric and D'Alembert's hypotheses predict that under all operational conditions the power loss of the outer raceway is higher than that of the inner raceway. This occurs because the spinning speed predicted with these models is higher in the outer contact, resulting in a greater power loss.

On the other hand, in the proposed model the power loss distribution of the inner and outer raceways varies depending on the operational conditions. At high angular velocities, the power loss of the outer raceway is higher than the inner, while at low angular velocities the opposite trend prevails. This pattern is the same for the low and high axial loads and it is directly related to the state of the spinning speed under each operational condition. At low angular velocities the spinning speed of the outer raceway is higher, causing higher power losses in this raceway. At high angular velocities the opposite is observed, i.e., the inner contact is subjected to higher spinning speeds, and thus higher power loss.

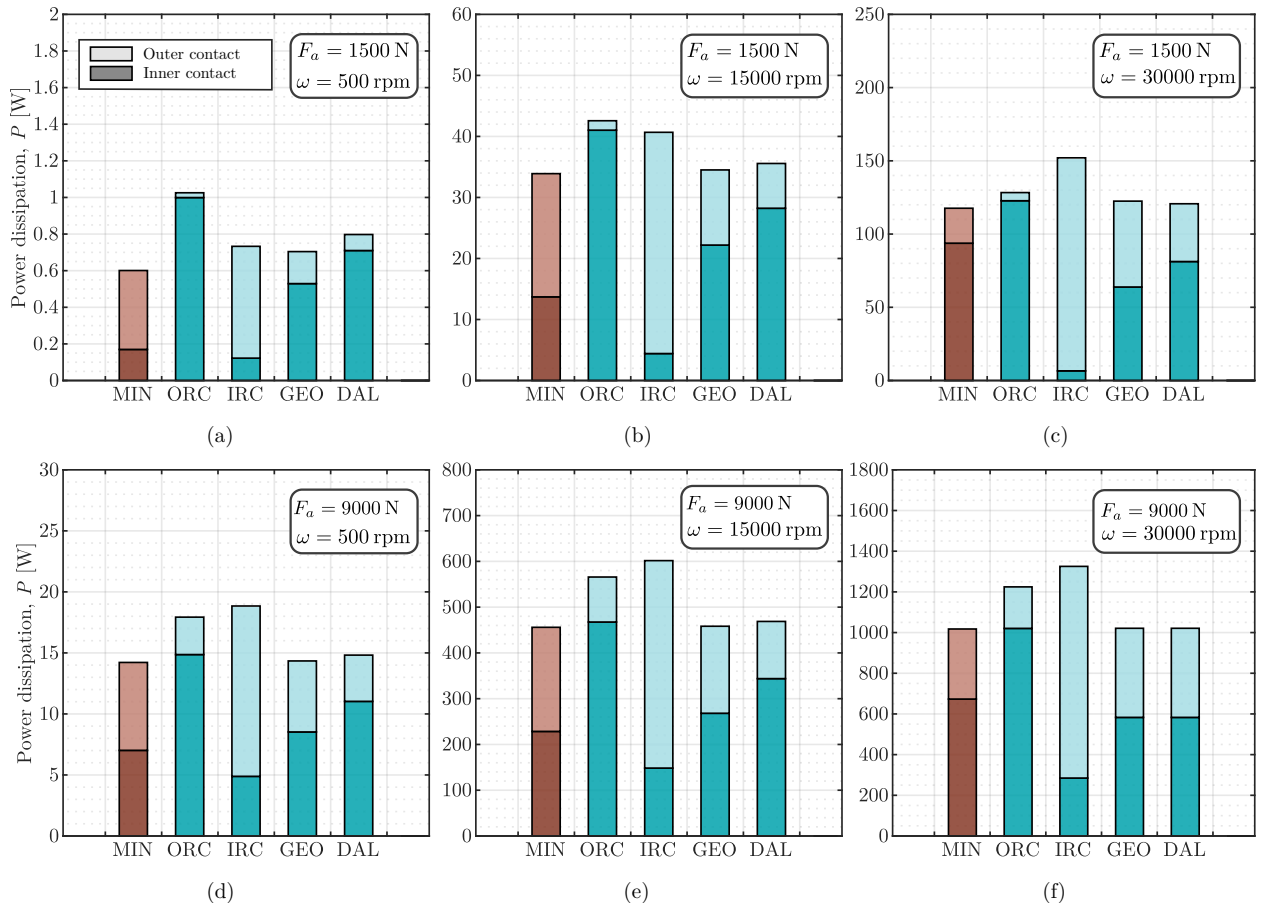


Figure 11: Comparison of power losses at the bearing component scale for different operational conditions calculated with the published kinematic hypotheses ($\Theta = 40^\circ\text{C}$).

In general, each hypothesis yields very different power loss predictions. The closest results to the proposed model are obtained with the geometric and D'Alembert's hypotheses. The deviations between the results are higher for independent predictions at the level of the inner and outer contact. In certain practical applications, it is more appropriate to predict the power loss of the inner and outer contacts independently, rather than focusing on the total power loss. For example, when determining the heat distribution between the inner and outer contacts, calculating the power loss of each contact independently allows for more detailed heat transfer analysis of the components. This analysis reveals that predictions can significantly vary, even by orders of magnitude, depending on the selected kinematic hypothesis and operating conditions.

5. Discussion

The results of this study have shown that the pitch angle calculated by the proposed model tends towards the IRC or the ORC hypothesis, depending on the operational conditions that the bearing is subjected to. In this section, the reasons for this are analysed from the point of view of lubrication performance. The discussion is centred on Fig. 10b, in which the influence of temperature on the pitch angle is plotted. Specifically, the trend of the pitch angle is analysed at 10000 rpm and 30000 rpm.

As the proposed model is based on the minimum power loss hypothesis, the behaviour of the pitch angle with respect to the operational conditions can be explained from this perspective. This means that the model tends towards the raceway control hypothesis which generates the lowest power loss. The plots in Fig. 12 illustrate the power loss predicted by the IRC and ORC hypothesis at various temperatures. At 10000 rpm (Fig. 12a), the power loss of the IRC is lower for all the simulated temperatures. Hence, the orientation of the pitch angle calculated with the proposed model is closer to the IRC. At 30000 rpm two trends are observed (Fig. 12b). At low temperatures, the IRC predictions yield lower power loss values, and the pitch angle tends to the IRC hypothesis. In contrast, at high contact temperatures the power loss of the ORC is lower and in consequence the pitch angle follows the trend of the ORC. This behaviour is in line with the results observed in Fig. 10b.

The power losses of lubricated contacts are governed by the rheology of the lubricant. Hence, to understand the trend of the power losses of each control hypothesis, the governing lubrication mechanism of the contact is analysed in the following paragraphs.

Fig. 12 highlights the lubrication mechanism that governs the contact interface of the ORC and IRC hypotheses at different temperatures. As described previously, almost all the power loss generated by the IRC or the ORC hypothesis occurs in the non-controlling raceway. In other words, the power loss is determined in the raceway in which the spinning motion occurs. Therefore, the governing lubrication mechanism of the ORC hypothesis is analysed only at the inner contact, and the IRC hypothesis at the outer contact. As stated in Eq. 37, the shear stress of any point Q of the contact interface is characterised either by the viscous behaviour of the lubricant or by the limiting shear stress (LSS). The contact ellipses of the figure show that the lubrication mechanism varies depending on the contact temperature and the angular velocity of the shaft:

- At high contact temperatures, the shear stress of the contact is governed by the viscous behaviour of the lubricant in all cases. The viscosity of the lubricant decreases exponentially when the temperature is increased. Under such conditions, the shear stress generated by the viscous term is lower than the LSS at all the points of the contact interface, and thus the viscous response governs the power loss of the contact.
- At moderate temperatures, the effect of the limiting shear stress appears. At both angular velocities, the effect of the LSS is higher along the inner contact, due to higher Hertzian pressures.
- At low contact temperatures, both contacts are governed mainly by the limiting shear stress behaviour of the lubricant. Under these conditions the viscosity of the lubricant is high, and as a result exceeds the threshold imposed by the LSS.

Thus, it can be concluded that the distribution of the governing lubrication mechanism varies significantly depending on the contact temperature. This mechanism affects the power loss of the contact, and hence the kinematic of the ball.

When the contact is governed by the viscous behaviour, the power loss of the ball is mainly determined by the contact pressure. The viscosity of the lubricant increases exponentially with pressure, and therefore high contact pressures generate higher shear stresses across the contact interface, resulting in higher power losses. This is particularly evident in the inner contact, in which the Hertzian pressure is higher, and hence the resultant power loss. In this case, the system reaches the minimum power loss state by reducing the spinning motion in the inner contact, and as a consequence the pitch angle tends towards the IRC hypothesis. This behaviour is observed at both low and high angular velocities. At high angular velocities the centrifugal forces increase the contact pressure

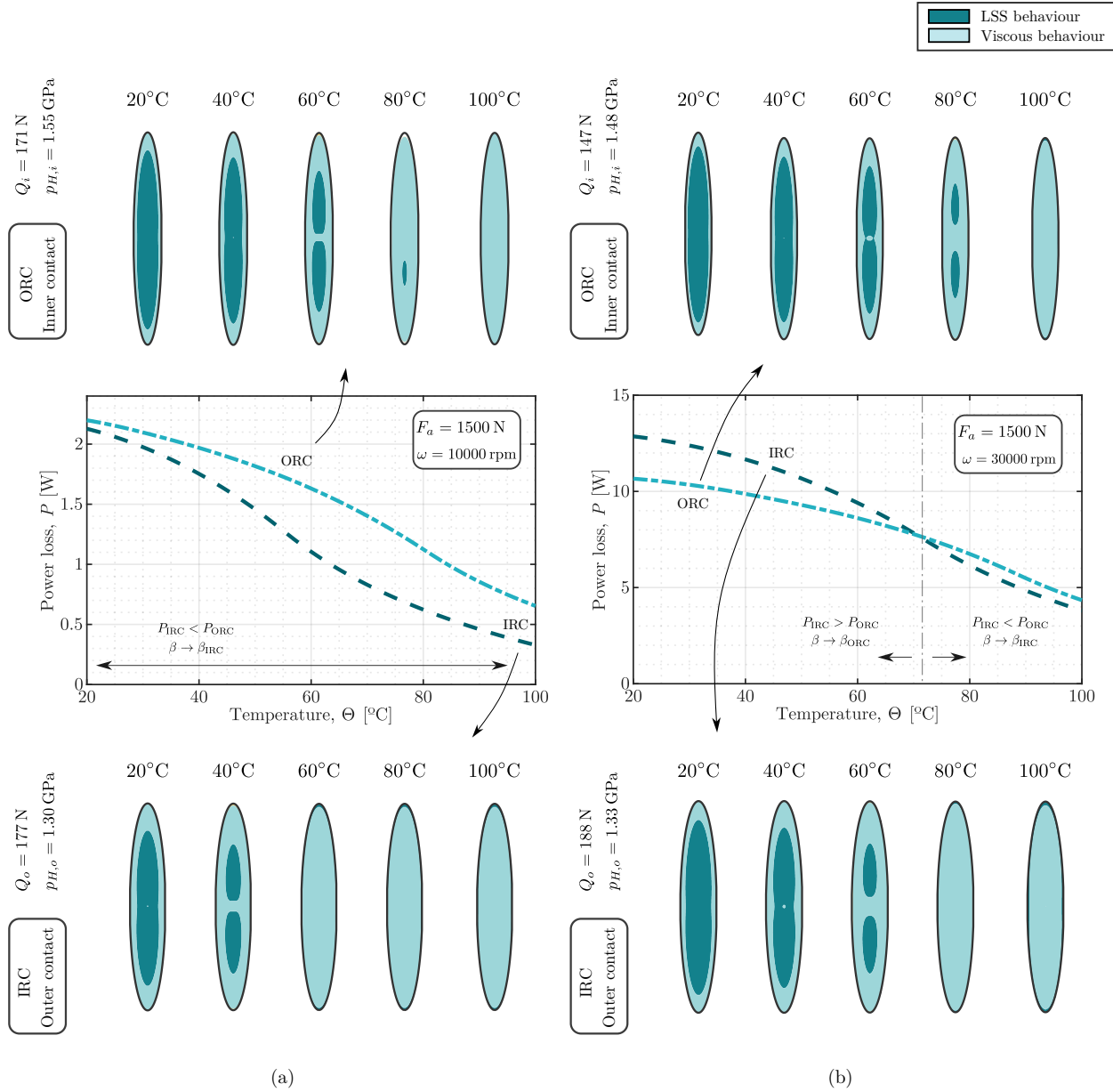


Figure 12: Governing lubrication mechanism of the contact for different contact temperatures, (a) at 10000 rpm and (b) at 30000 rpm ($F_a = 1500$ N).

of the outer raceway, but the inner contact pressure is still higher. Therefore, when the viscous response is the governing mechanism, the influence of the centrifugal forces is minimal.

On the other hand, when the contact is governed by the LSS behaviour, the shear stress is determined by the limiting stress-pressure coefficient. Since this variable is constant, the pressure has a negligible effect on the total power loss of the contact. Under such conditions, the reaction force is the variable that governs the power loss, instead of the contact pressure. At low angular velocities, the centrifugal forces are low, and the contact load of the inner and outer raceway are similar. Under such conditions the contact surface is not completely governed by the LSS behaviour and a proportion of the surface is characterised by the viscous response. In this case, the proportion of the viscous response along the outer contact is higher and the power loss of the inner contact is also higher as a result. To minimize the power loss, the ball tends to reduce the spinning speed along the inner contact and the pitch angle approaches the IRC hypothesis. In contrast, at high angular velocities the centrifugal forces increase, generating higher reaction forces, and thus greater power losses, in the outer raceway. Consequently, the minimum power loss is obtained by the reduction of the spinning speed of the outer contact, and hence the kinematic of the contact is closer to the ORC hypothesis.

6. Conclusions

In this paper, a minimum energy based quasi-static model is presented to determine the kinematic of lubricated contacts in ball bearings. The results of this study have given rise to the following conclusions:

- (1) A generalised ball bearing quasi-static model which is based on the minimum power loss hypothesis was developed. The power loss of the contact is calculated in the presented model by introducing a theoretical EHL model which considers local effects produced by spinning motion. The contact kinematic is solved with frictional interactions, rather than relying on simplified hypotheses that do not consider these phenomena.
- (2) The orientation of the pitch angle governs the spinning motion of both the inner and outer contacts, significantly influencing the sliding speed distribution. This, in turn, determines the power loss distribution of each contact and the total power loss of the ball. Consequently, the main parameter that determines the power loss of the ball is the pitch angle.
- (3) The pitch angle is highly sensitive to the operational conditions that the bearing is subjected to. The results of this study were compared to other published hypotheses, and it was observed that at low angular velocities, the predictions of the proposed model closely align with the IRC hypothesis, while at high angular velocities it approximates the ORC hypothesis. This transitional behaviour confirms the prevailing consensus in the field, and is supported by experimental evidence found in the literature. The axial load and the temperature of the lubricant also have a determinant influence on the kinematic of the contact, particularly at high angular velocities. In contrast, the geometric and D'Alembert's hypotheses do not predict the observed transitional behaviour. However, they do consider a combined rolling and spinning motion in both ball-raceway contacts, which is a similar kinematic state to that predicted by the proposed model.
- (4) The power loss predictions obtained with each kinematic hypothesis deliver very different results at the bearing component level. Generally, when considering the overall power losses of the entire component, the geometric and D'Alembert's hypotheses closely align with the results obtained from the proposed model. The predictions conducted with these two hypotheses, fit relatively well with the minimum energy approximation. However, the deviation of the results obtained with the raceway control hypotheses are greater. In this case, the predicted power loss is much higher than that calculated by the proposed model. These differences are more pronounced when the power loss of the inner and outer contacts is analysed independently. Under certain operational conditions, predictions from each kinematic hypothesis yield results of varying orders of magnitude.
- (5) The rheology of the lubricant plays a crucial role in the kinematic behaviour of the contact. The combination of the governing lubrication mechanism of the contact and the influence of centrifugal forces are key variables that determine the orientation of the pitch angle. The kinematics of the contact will vary depending on the lubricant used. This suggests that theoretically, it would be possible to achieve the necessary kinematic conditions by using a specific lubricant.

The discussion of the results is conducted considering pure axial loads. Future investigations might usefully analyse the kinematic and power loss behaviour of the balls under combined axial and radial loads. In such conditions, the balls are not uniformly loaded and hence their kinematic and power loss state changes depending on the azimuth position in which they are located.

One of the main advantages of the model presented here is that it is composed of independent modules. This facilitates the incorporation of novel features into the EHL model, to more accurately characterize the frictional interactions of the contact. Further investigations might usefully explore integrating more sophisticated EHL models which consider factors such as new lubricants, thermal effects, or mixed lubrication phenomena, all without altering the general structure of the quasi-static model.

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Appendix A. Pitch angle calculation models from literature

A number of kinematic hypothesis to calculate the motion of the ball can be found in the scientific literature. The pitch angle expression of these models is summarised hereafter.

A1. Inner raceway control hypothesis

This hypothesis determines the pitch angle in such a way that the all the spinning motion is given in the outer raceway [18]:

$$\beta = \text{atan} \left(\frac{\sin \alpha_i}{\cos \alpha_i - \gamma'} \right) \quad (\text{A.1})$$

where, $\gamma' = D/d_m$.

A2. Outer raceway control hypothesis

This hypothesis defines the pitch angle such that all the spinning motion occurs in the inner raceway [18].

$$\beta = \text{atan} \left(\frac{\sin \alpha_o}{\cos \alpha_o + \gamma'} \right) \quad (\text{A.2})$$

A3. D'Alembert's hypothesis

This hypothesis is based on the D'Alembert's inertia force principle. It assumes that the power conducted by all the reaction forces of the ball is zero [44, 20].

$$\beta = \text{atan} \left(\frac{C \left(\frac{1+\gamma' \cos \alpha_o}{1-\gamma' \cos \alpha_i} + 1 \right) \sin \alpha_i + 2 \sin \alpha_o}{C \left(\frac{1+\gamma' \cos \alpha_o}{1-\gamma' \cos \alpha_i} + 1 \right) \cos \alpha_i + 2 (\cos \alpha_o + \gamma') + \gamma' C \left(\cos (\alpha_i - \alpha_o) - \frac{1+\gamma' \cos \alpha_o}{1-\gamma' \cos \alpha_i} \right)} \right) \quad (\text{A.3})$$

in which C is the ratio of spinning friction moment for the inner and outer raceway:

$$C = \frac{W_o a_{H,o} L_2(k_{e,o})}{W_i a_{H,i} L_2(k_{e,i})} \quad (\text{A.4})$$

with $L_2(k_e)$ representing the second kind of elliptic integral and k_e the ellipticity ratio of the contact.

A4. Geometric hypothesis

This hypothesis assumes that the orientation of the pitch angle is equal to the mean value of the inner and outer contact angles [47]:

$$\beta = \frac{\alpha_i + \alpha_o}{2} \quad (\text{A.5})$$

A5. Solution procedure

Fig. A.1 illustrates the solution procedure followed by the quasi-static literature models defined above. The governing steps are similar to those introduced in Fig. 5. The main difference between these models and the proposed model is the step related to the calculation of the pitch angle. The equations introduced in this appendix Eqs. (A.1)-(A.5) are employed to calculate the pitch angle in this step. The expression selected depends on the hypothesis followed. The flowchart clearly shows that the models from the literature do not calculate the pitch angle considering power losses generated by the lubricated effects. These variables are calculated once the pitch angle is determined. Therefore, these models cannot consider the effect of the lubricant rheology in the kinematic of the ball. It should be noted that all of these hypotheses initialise the pure rolling radii considering it as equal to the radii of the ball ($r'_i = r'_o = D/2$). The diagram shows that the solution procedure does not recalculate this variable, and thus these literature hypotheses assume that the pure rolling radii is the radii of the ball.

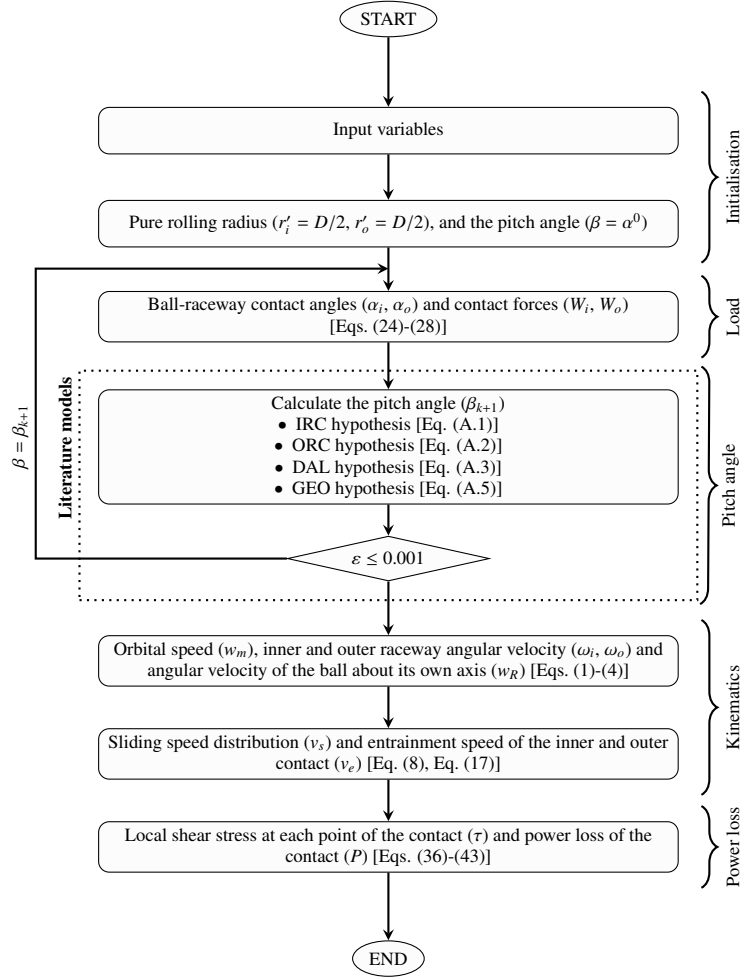


Figure A.1: Flowchart of the literature models.

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